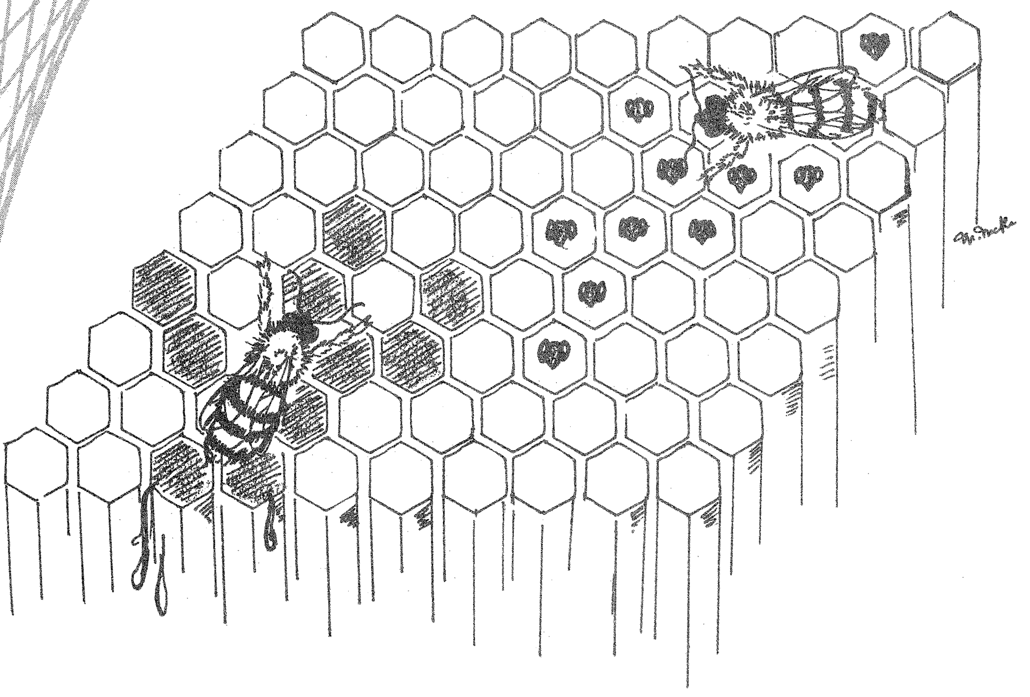
MATHEMATICS MAGAZINE



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COMPUTERS & GROUPS • ISOMETRIES
LEPIDOPTERA • HEX BY INDUCTION

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COVER: A simple inductive proof (see page 85) shows that in a full hive there must either be a row of honey cells from one side to the other or a row of pupae from top to bottom. So without even knowing the rules, one of these bees is bound to produce a "winning" chain in beehive Hex.

ARTICLES

NOTES

PROBLEMS

NEWS AND LETTERS

EDITORIAL POLICY

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Authors planning to submit manuscripts may find it helpful to obtain the more detailed statement of guidelines available from the editorial office.

ABOUT OUR AUTHORS

Joseph Gallian ("Computers in Group Theory") is a graduate of Slippery Rock State College and holds a Ph.D. in mathematics from Notre Dame. His interest in utilizing computers to illustrate the theory of finite groups arose from undergraduate research projects at the University of Minnesota at Duluth, where he is presently. Gallian's concern with bringing group theory to undergraduates has resulted in an article on the history of finite groups (to appear in a subsequent issue) and a presentation on letter sorting (group theory for post office officials) which he made at a regional M.A.A. meeting. He has also published a number of research papers on finite groups.

Donald Greenwell and Peter D. Johnson ("Functions that Preserve Unit Distance") met at Emory University in the fall of 1973 and among the problems they traded was the one which produced this article. Johnson had originally come across problems of this sort while practicing for the Putnam exam at Michigan State. After about two months they produced this proof which Johnson found "shocking". Currently Greenwell is an N.R.C. Research Associate at Marshall Space Flight Center and Johnson is at the American University of Beirut, Lebanon.

Jeffrey Jaffe ("Permutation Numbers") is an undergraduate at MIT. Last year while taking a course in combinatorics, he arranged to work with the teacher, Professor Richard P. Stanley, in conjunction with MIT's Undergraduate Research Opportunities Program. His paper on permutation numbers was one of the results.

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Computers in Group Theory

A report on several student projects taking advantage of the number theoretic aspects of many theorems about finite groups.

JOSEPH A. GALLIAN
University of Minnesota, Duluth

Much of finite group theory is arithmetical in nature and thus lends itself to computer analysis. In this paper, we describe some computer related projects in group theory which have been done by undergraduates at the University of Minnesota, Duluth, during the past two years. We believe that projects of this kind provide valuable learning experiences for students.

The projects we discuss concern two very important classes of finite groups — Abelian groups and simple groups. A simple group is a nonabelian group whose only normal subgroups are the identity and the group itself. Thus, as far as the existence of normal subgroups is concerned, Abelian groups and simple groups are at opposite extremes since every subgroup of an Abelian group is normal, while no subgroup (excluding the two trivial cases) of a simple group is normal.

Let's consider Abelian groups first. Utilizing the Fundamental Theorem of finite Abelian groups [**13**, Sec. 2.14] a program was written (all programs were written in Fortran) to determine the isomorphism classes of all Abelian groups of a given order. For example, for the integer 133128 the computer prints out the twelve isomorphism classes of groups of this order as direct products of cyclic groups such as:

$$C(43)\times C(43)\times C(3)\times C(3)\times C(2)\times C(2)\times C(2).$$

Here $C(k)$ denotes the cyclic group of order k.

Furthermore, a tabulation was kept to determine how many integers in a given interval give rise to a unique Abelian group (e.g., 15, 501), how many give rise to two Abelian groups (e.g., 28, 522) and so on. The results obtained were quite unexpected. TABLE 1 indicates the various percentages for several arbitrarily chosen intervals of various sizes. We see that in the interval 1–10,000, 60.83% of the integers correspond to unique Abelian groups, 20.08% correspond to two Abelian groups and so on. The surprising fact, of course, is that these percentages are to a remarkable extent independent of the interval chosen.

We next describe a project involving Abelian groups and number theory. For each positive integer n, let $U(n)$ denote the set of all positive integers less than or equal to n and relatively prime to n. Then $U(n)$ is an Abelian group (under multiplication modulo n) of order

$$\phi(n)=\prod_{i=1}^{k} p_i^{\alpha_i}-p_i^{\alpha_i-1} \qquad \text{where} \qquad n=\prod_{i=1}^{k} p_i^{\alpha_i}$$

[**17**, p. 61]. Let us call such groups U-groups. A program was written to determine each of the following for any positive integer n: the multiplication table for $U(n)$, the isomorphism class of $U(n)$

No. of groups	1 – 10000	50001 – 50500	500001 – 505000	500001 – 550000	900001 – 901000	999001 – 1000000
	Intervals					
1	60.83	61.00	60.82	60.80	60.90	60.80
2	20.08	19.80	20.04	20.06	19.90	19.70
3	7.44	7.20	7.38	7.42	7.70	7.70
4	2.20	2.20	2.22	2.23	2.40	2.30
5	3.21	3.40	3.20	3.20	3.00	3.20
6	1.46	2.00	1.42	1.45	1.20	1.60
7	1.51	1.60	1.50	1.47	1.60	1.40
8	0.08	0.00	0.14	0.10	0.10	0.10
9	0.22	0.20	0.28	0.22	0.30	0.10
10 or more	2.97	2.60	3.00	3.05	2.90	3.10

Percentages of integers in given ranges that are the order of the given number of distinct Abelian groups.

TABLE 1

(in the form $C(k_1)\times C(k_2)\times\cdots\times C(k_j)$), the order of each element of $U(n)$, the exponent of $U(n)$ and, up to isomorphism, all subgroups of $U(n)$.

It is easily shown that for any given integer N there may be several nonisomorphic U-groups of order N or none at all. In fact, it follows from the formula above that no odd integer greater than 1 is the order of a U-group and there are many even integers with the same property (e.g., 26, 50 and 98). Thus the opposite question of which Abelian groups are realizable as U-groups naturally comes to mind. A program which achieves the following was also written: Given an integer N, the computer finds all solutions of the equation $\phi(x)=N$ (see **[5**, p. 82]) where x has at most 5 distinct prime divisors (this restriction was made to simplify the programming, but no solutions are missed if $N<5760$). If the solutions $x_1, x_2, \ldots, x_j$ are found, the machine then determines the distinct isomorphism classes of the groups $U(x_1), \ldots, U(x_j)$ and prints out each one as a direct product of cyclic groups. For example, by taking $N=32$ the following is obtained:

$$C(2)\times C(16)=U(64)=U(68)=U(102)=U(51)$$

$$C(2)\times C(2)\times C(8)=U(96)$$

$$C(2)\times C(4)\times C(4)=U(80)$$

$$C(2)\times C(2)\times C(2)\times C(4)=U(120).$$

Thus we see that there are four integers whose corresponding U-groups are models of $C(2)\times C(16)$ but no U-group is isomorphic to $C(32)$ or $C(8)\times C(4)$ or $C(2)\times C(2)\times C(2)\times C(2)\times C(2)$. (It is true however that every Abelian group is isomorphic to a subgroup of some U-group **[17**, p. 96]).

We now turn to the project in simple group theory. At present, simple group theory is the most active and glamorous area of research in the theory of groups. In many respects, the concentration of intellectual effort and resulting deep discoveries in simple group theory during the 1960's and 1970's is comparable to that found in atomic theory during the 1920's and 1930's. The principal reason for focusing on simple groups is that they (together with the cyclic groups of prime order) play a role in group theory somewhat analogous to that which the primes play in number theory or the elements do in chemistry or the amino acids do in the study of protein. That is, they serve as the "building blocks" for all groups. We emphasize that the adjective "simple" does not connote the complexity of the structure of these groups but merely the absence of normal subgroups. Indeed, for many problems about finite groups the most difficult cases are those involving the simple groups. (See **[8]** for a nontechnical history of simple group theory.)

The goal of this project is to determine which integers are and which are not the orders of simple groups. First, 35 theorems which could possibly determine whether a group of a given order is not simple were programmed. These theorems, which are listed in the Appendix, require no structural assumptions on the groups other than order. To illustrate, consider Theorem 3: *If for some* $p \mid N$ *the only integer* m *that divides* N *and is congruent to* 1 mod p *is* 1, *then* N *is not the order of a simple group.* Thus given an integer N the computer merely finds all integers m which have the first two properties. If 1 is the only such integer for some p then any group of order N is not simple. This test, although very simple to apply, is extremely powerful. (Group theorists will note that the number of Sylow p-subgroups of a group of order N must satisfy the two conditions in Theorem 3; the computer, of course, doesn't know this, nor does it need to.)

A program was written to check each integer in a given interval against these theorems to determine whether the integer is not the order of a simple group. The output is illustrated in TABLE 2.

N	*Theorems*														
⋮															
1678	3	4	5	7	8	10	13	14	16	18	32	33	34		
1679	3	4	5	8	10	11	14	17	18	28	30	32	33	34	35
1680	8														
1681	1	3	4	5	8	10	11	14	17	18	20	28	30	33	34
1682	2	3	4	5	7	8	10	13	14	18	32	33	34		
⋮															

A list of the theorems on simple groups that prove, for a given N, that N is not the order of a simple group.

TABLE 2

Thus we see that each of theorems 3, 4, 5, 7, 8, 10, 13, 14, 16, 18, 32, 33 and 34 proves that there is no simple group of order 1678. On the other hand, only theorem 8 eliminates the possibility of the existence of a simple group of order 1680.

Also, a tally was kept on how many integers each theorem ruled out. In this way, there is obtained for each theorem a qualitative measure of its efficiency for determining which integers are not the orders of simple groups. For example, Theorem 3 (Sylow's theorem) eliminates 9,431 of the first 10,000 integers and 9,770 of the integers between 100,001 and 110,000 as possible orders for simple groups. Thus one might say that for the integers from 1 to 10,000 Sylow's theorem is a "94% theorem" while in the interval 100,001 to 110,000 it is a "97% theorem". Similarly, Theorem 4 could be called a "78% theorem" in the interval 1–10,000.

Altogether, the theorems programmed ruled out all but 16 of the first 10,000 integers. Since it is known **[12]** that there are exactly 16 integers between 1 and 10,000 which are orders of simple groups we observe that, in this range, the project is 100% efficient.

Finally we turn our attention to the integers which are the orders of simple groups. In 1901 L. E. Dickson gave a list **[6**, p. 309**]** of all the orders of the known simple groups as far as one million (53 in all) and a partial list of the orders of the known simple groups up to one billion. Although numerous new simple groups of "low order" have recently been discovered, no complete update of Dickson's list has appeared in print. Thus a program was written to produce a list of the orders of all the known simple groups as far as 100 trillion.

With a handful of exceptions (fewer than 26) the known finite simple groups can be placed into 17 infinite families according to their method of construction and corresponding to each family there is a

1.	$\frac{1}{d} q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1}-1)$	$d=(n+1,q-1)$	7.	$q^2(q^2+1)(q-1)$
2.	$\frac{1}{d} q^{n^2} \prod_{i=1}^{n} (q^{2i}-1)$	$d=(2,q-1)$	8.	$q^6(q^6-1)(q^2-1)$
3.	$\frac{1}{d} q^{n^2} \prod_{i=1}^{n} (q^{2i}-1)$	$d=(2,q-1)$	9.	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
4.	$\frac{1}{d} q^{n(n-1)}(q^n-1) \prod_{i=1}^{n-1} (q^{2i}-1)$	$d=(4,q^n-1)$	10.	$q^3(q^3+1)(q-1)$
5.	$\frac{1}{d} q^{n(n-1)}(q^n+1) \prod_{i=1}^{n-1} (q^{2i}-1)$	$d=(4,q^n+1)$	11.	$\frac{n!}{2}$
6.	$\frac{1}{d} q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1}-(-1)^{i+1})$	$d=(n+1,q+1)$		

Formulas for the orders as far as 100 trillion of the known finite (nonsporadic) simple groups, with certain elementary exclusions omitted.

TABLE 3

formula which yields the orders of the groups (see [**9**, pp. 491–492] for details). Since six of the formulas render no integers less than 100,000,000,000,000 only 11 were programed; they are listed in TABLE 3. Formula 11, for instance, represents the orders of the simple alternating groups of even permutations, while formula 1 represents the orders of the simple groups obtained by considering the square matrices of size $n+1$ and determinant 1 with entries from the Galois field with q elements modulo the subgroup of scalar matrices (this subgroup has order d).

The 25 or so "missing" groups mentioned above are called "sporadic" because they are constructed via *ad hoc* methods which do not produce an infinite family. Interestingly, the demonstration of the very existence of several of the sporadic groups requires the use of computers. (A fascinating account of how many of the sporadic groups were discovered is given in [**10**, Sec. I. 3]). Since 17 of these groups have order under 100 trillion the list is completed by merely inserting these integers in the appropriate places.

We conclude by remarking that the ideas discussed above by no means exhaust the possibilities for undergraduate student projects of this nature and we urge teachers of abstract algebra to encourage their students to undertake similar endeavors.

The programming mentioned in this paper was done by the following students: Paul Telega, Craig Cato, Jon Anderson, Jeffrey Stein, Chris Hull, Ronald Solyntjes, John Smith, Jay Boekhoff and Mark Bolf. The author acknowledges their contribution to the paper and thanks them for their interest and cooperation in these projects. I am also indebted to the editors for a number of valuable suggestions pertaining to organization and style of the paper.

Appendix

Each of the 35 theorems used to determine integers that cannot be the order of a simple group has the form "If P, then N is not the order of a simple group," where P is a number-theoretical statement about the integer N. We list below the propositions P, with reference to the associated group-theoretic theorem. In each statement, $p, p_1, p_2, \ldots, p_j$, and q represent distinct primes.

1. $N = p^k$. [**13**, p. 86]
2. For some p such that $p^2 | N$ and $p^3 \nmid N$, $N \leq p^5$. [**7**]
3. For some $p | N$ the only integer m that divides N and is congruent to 1 mod p is 1. (Sylow [**13**, sec. 2.12])
4. $N = p_1^{k_1} p_2^{k_2} \ldots p_j^{k_j}$ and $N \nmid (N/p_i^{k_i})!$ for some i. [**13**, p. 73]
5. $12 \nmid N$ and $320 \nmid N$. [**9**, p. 259]
6. $N = 16m$, $N < 10^6$, $15 \nmid N$, $21 \nmid N$, $2448 \nmid N$, $5616 \nmid N$, $6072 \nmid N$, $N \neq 29120, 32736, 51888$. [**11**, p. 147]
7. $N = 2k$, k odd. [**16**, p. 138]

8. $N = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, $k_1 + k_2 + \cdots + k_j < 10$, $N \neq \frac{1}{2}p^r(p^{2r}-1)$ $(p \geqq 3)$, $N \neq 2^r(2^{2r}-1)$ $(r > 1)$, $N \neq 2520$, 5616, 7920, 175560, 6048, 29120. [**1,** p. 115]
9. For some $p \mid N$ where $p^2 \nmid N$ and for all $m > 1$ where $m \equiv 1 \bmod p$ $N/m = pq$ where q is prime and $\min\{p, q\} \nmid (\max\{p, q\} - 1)$. [**16,** Th. 6.2.9, p. 137] and [**13,** pr. 11a, p. 102]
10. $N = p^i q^j$. (Burnside [**9,** p. 131])
11. $N = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, $k_1 = 1$ or 2 and $2 < p_1 < p_2 < \cdots < p_j$. [**16,** p. 141]
12. $59 \mid N$, $N < 10^6$, $N \neq 102660$. [**11,** p. 150]
13. $2 \mid N$, $8 \nmid N$ and $12 \nmid N$. [**11,** p. 142]
14. For some $p \mid N$, $p^3 > N$, $N \neq \frac{1}{2}p(p^2-1)$ $(p > 3)$, $N \neq 2^r(2^{2r}-1)$ where $2^r + 1$ is prime. [**11,** p. 142]
15. $8 \mid N$, $16 \nmid N$, $105 \nmid N$, $273 \nmid N$, $N < 10^6$, $N \neq 168$, 360, 504, 6072, 7800, 194472, 352440, 546312, 885720. [**11,** p. 147]
16. $p \mid N$, $p > 97$, $N < 10^6$, $N \neq 515100$, 546312, 612468, 647460, 721392. [**11,** p. 148]
17. $2 \nmid N$. (Feit-Thompson [**9,** p. 450])
18. $12 \nmid N$, $16 \nmid N$, $56 \nmid N$. (Burnside and Feit-Thompson [**16,** p. 399])
19. $N = 2^a 3^b p$, $a > 0$, $b > 0$, $N \neq 60$, 168, 360, 504, 2448, 5616, 6048, 25920. [**20,** p. 124]
20. $N = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, $N < 10^6$, and for some i, $p_i \geqq 37$ with $k_i \geqq 2$. [**11,** p. 148]
21. $N = p_1 p_2^b p_3^c$, $p_1 > p_2^b > p_3^c$, $c > 0$, $N \neq 60$, 504, 2448. [**14,** p. 383]
22. $N = p_1^a p_2 p_3^c$, $p_1^a > p_2 > p_3^c$, $c > 0$, $N \neq 168$. [**14,** p. 383]
23. $4 \mid N$, $8 \nmid N$, $N \neq \frac{1}{2}p^r(p^{2r}-1)$ $(p \geqq 3)$. [**11,** p. 145]
24. $N = 2^a 3 p^b$, $a > 0$, $b > 0$, $N \neq 60$, 168, 504, 2448. [**15,** p. 116]
25. $N = 2^a 3^2 5^b$, $N < 10^6$, $a > 0$, $b > 0$, $N \neq 360$. [**12,** p. 101]
26. $N = 2^a 3^2 7^b$, $N < 10^6$, $N \neq 504$, $a > 0$, $b > 0$. [**12,** p. 101]
27. $N = p_1^a p_2^b p_3^c$, $a > 0$, $b > 0$, $c > 0$, $60 \nmid N$, $168 \nmid N$, $504 \nmid N$, $2448 \nmid N$, $5616 \nmid N$. [**18,** p. 388]
28. $N = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, $2 \nmid N$, $k_1 + \cdots + k_j < 7$. [**2,** p. 268]
29. $N = p_1^{k_1} \cdots p_j^{k_j}$, $2 \nmid N$, $k_i = 1$ for some i and $p_i = 2^n + 1$ for some n. [**2,** p. 262]
30. $N = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, $2 < p_1 < p_2 < \cdots < p_j$, with $k_1 = 1$ or 2 or with $k_1 = 3$ and $(N, p_1^2 + p_1 + 1) = 1$. [**2,** p. 258]
31. $N = p3^a2^b$, $a > 0$, $b > 0$, and it is not true that $1/p \leqq 2^b/3^a \leqq p$. [**19,** p. 190]
32. For some p such that $p^n \mid N$ and $p^{n+1} \nmid N$, $k = \Pi_{i=1}^n (p^i - 1)$ and $r = N/p^n$ $(r > 1)$ are relatively prime. [**16,** p. 399]
33. $N = p^a q^b g$, $a > 0$, $g - 1 < \log p / \log 6$ (g need not be prime). [**3,** p. 349]
34. $3 \nmid N$, $N < 10^6$, $N \neq 29120$. [**11,** p. 147]
35. $N = q^i pm$, $i > 0$, $m < p - 1$, $N < 10^6$, $N \neq 60$, 168, 2448, 4080, 14880. [**4,** p. 758]

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Functions that Preserve Unit Distance

Mappings need only preserve certain distances in order to be congruences.

DONALD GREENWELL
Marshall Space Flight Center

PETER D. JOHNSON
American University of Beirut

1. Introduction

Functions which preserve distance (isometries, congruences) lie at the heart of Euclidean geometry. In this paper we study a number of relatively weak conditions which are sufficient to guarantee that a transformation is distance preserving. For example Beckman and Quarles [**1**], and then independently Bishop [**2**], and in a special case Zvengrowski [**3**, Appendix to Chapter II] discovered that *if $n \geqq 2$ and T: $E^n \to E^n$ is a mapping which preserves unit distances, then T preserves all distances.* We present here a number of similar results which assert that preservation of a fixed distance in some, but not all, directions is (or, in some cases, is not) sufficient to guarantee that a transformation be an isometry. More precisely, if we consider a "direction" in E^n to be a point on the unit sphere we discuss in what sets of directions it is sufficient for a transformation to preserve unit distances in order that it be an isometry.

2. A necessary condition

Let E^n denote n-dimensional real Euclidean space, $d(\ ,\ )$ the usual Euclidean metric, and S^{n-1} the unit $n-1$ sphere in E^n. If $D \subseteq S^{n-1}$ and T: $E^n \to E^n$, we shall say that T **preserves the distance r in directions** D if and only if $d(u, v) = r$ and $(u-v)/r \in D$ imply $d(T(u), T(v)) = r$. Note that if T preserves the distance r in directions D_1 and D_2, then T preserves the distance r in directions $-D_1$ and $D_1 \cup D_2$.

In this section we take up the question of what subsets D of S^{n-1} have the property that the only mappings of E^n into E^n which preserve the distance 1 in directions D are isometries. The result of Beckman and Quarles states that, for $n \geqq 2$, S^{n-1} itself has this property.

For $D \subseteq S^{n-1}$, let $A(D)$ denote the additive subgroup of $(E^n, +)$ generated by D, and T a mapping from E^n to E^n. We consider the following properties of D:

$P1$: If $u-v \in D$ implies $T(u)-T(v) \in S^{n-1}$, then T is an isometry.
$P2$: If $u-v \in D$ implies $T(u)-T(v) \in A(D) \cap S^{n-1}$, then T is an isometry.
$P3$: If $u-v \in D$ implies $T(u)-T(v) \in D$, then T is an isometry.
$P4$: If $u-v \in D$ implies $T(u)-T(v) = u-v$, then T is an isometry.

Property $P1$ is just the property we are investigating; namely, that the only mappings from E^n to E^n that preserve the distance 1 in directions D are isometries. Clearly property Pk implies property $P(k+1)$, for $k = 1, 2$ or 3.

THEOREM 1. *$D \subseteq S^{n-1}$ has property $P4$ if and only if $A(D) = E^n$.*

Proof. Suppose $A(D) \neq E^n$. Take some $w \in E^n \backslash A(D)$ and define T: $E^n \to E^n$ by $T(u) = u$ if $u \in A(D)$, $u + w$ if $u \notin A(D)$. Clearly, if $u - v \in D$, then $T(u) - T(v) = u - v$, yet T is not an isometry, since $d(T(w), 0) \neq d(w, 0)$. Thus $A(D) \neq E^n$ implies D does not satisfy $P4$.

Now suppose $A(D) = E^n$. Then each $u \in E^n$ can be represented as a finite sum $\Sigma m_\alpha w_\alpha$, with each m_α an integer and each w_α an element of D. Let $\|u\| = \min \Sigma |m_\alpha|$, with the minimum being taken over all such representations of u. Now suppose T: $E^n \to E^n$ satisfies the condition that if $u - v \in D$, then $T(u)-T(v)=u-v$. It is easy to prove, by induction on m, the following statement: if $\|u\|=m$, then $T(u)=u+T(0)$. Thus $T(u)=u+T(0)$ for all $u \in E^n$, so T is an isometry. Therefore, T satisfies $P4$.

COROLLARY 1.1. *If D has property $P1$, then $A(D)=E^n$.*

COROLLARY 1.2. *$P1$ and $P2$ are equivalent.*

Proof. We already know that $P1$ implies $P2$.

If $P2$ holds, then $P4$ holds, so $A(D)=E^n$. Then $A(D)\cap S^{n-1}=S^{n-1}$, so statement $P2$ is just statement $P1$.

COROLLARY 1.3. *If $|D|<2^{\aleph_0}$, then D does not have property $P1$ (or properties $P2$, $P3$, or $P4$ for that matter).*

Thus, there is no countable set of directions D such that a mapping preserving the distance 1 in those directions is forced to be an isometry.

It may be of some interest to note that, by Theorem 1, and by the methods used in its proof, the following three statements about a set $D\subseteq S^{n-1}$ are also equivalent to $P4$:

(i) If $u-v\in D$ implies $T(u)-T(v)=u-v$, then T is a translation;

(ii) If $u-v\in A(D)$ implies $T(u)-T(v)=u-v$, then T is an isometry;

(iii) If $u-v\in A(D)$ implies $T(u)-T(v)=u-v$, then T is a translation.

We shall make no use of the equivalence of these statements. They are offered only for the reader's delectation.

3. A sufficient condition

We would like to know if the converse of Corollary 1.1 is true, but have no idea, not even a guess, at present. Corollary 1.1 gives a necessary condition for D to have property $P1$. Theorem 2 gives a sufficient condition.

THEOREM 2. *If $n\geqq 2$ and $|S^{n-1}\backslash D|<2^{\aleph_0}$, then every mapping from E^n to E^n which preserves the distance 1 in directions D must preserve all distances.*

The skeleton of the proof and much of the flesh comes from Bishop's paper. We shall give Bishop's sequence of preliminary lemmas, in the sharpened form necessary to our purpose, and the full proof in the case $n=2$, leaving the case $n>2$ to those readers who will familiarize themselves with Bishop's paper and who have caught the spirit of the modifications of his arguments given here.

In what follows, T will stand for a mapping from E^n to E^n, $n\geqq 2$, and $D\subseteq S^{n-1}$ will satisfy the hypothesis of Theorem 2; i.e., $|S^{n-1}\backslash D|<2^{\aleph_0}$. Henceforward, any set of directions D satisfying this requirement will be called a large set of directions.

LEMMA 2.1. *If T preserves the distance $r>0$ in a large set of directions D, and if $d(u,v)^{-1}(u-v)\in D$, and $(k-1)r<d(u,v)\leqq kr$ for some integer $k\geqq 3$, then $d(T(u),T(v))\leqq kr$.*

Proof. If u and v were joined by a polygonal path consisting of k segments, each of length r, and each pointing in a direction in D, then the conclusion would follow.

If $d(u,v)=kr$, there is clearly no difficulty in connecting u and v by such a path.

Suppose $(k-1)r<d(u,v)<kr$. Let P be a plane containing u and v (if $n=2$, there is only one plane that will do; otherwise, there is a great deal of freedom at this stage). For θ sufficiently small we can lay off $k-2$ segments of length r, starting from u, on a straight line in P making an angle θ with the segment uv, and still reach v from the end of the last of these segments by adjoining two more segments of length r, in P (see FIGURE 1).

Furthermore, there are only two possible ways of adjoining the final two segments. We choose the one which does not cross the segment uv.

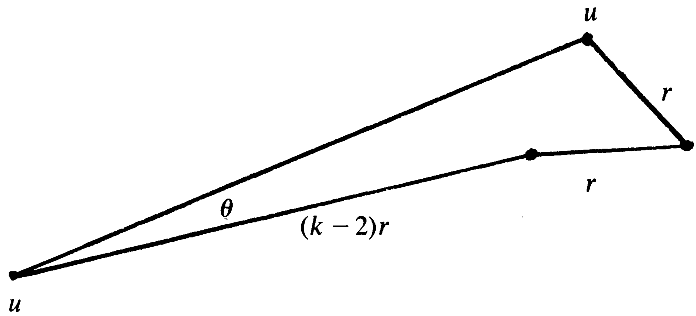

FIGURE 1

In the resulting polygonal path of k segments, each of length r, from u to v, there are three directions represented. Furthermore the correspondence between θ and each of these three directions is one-to-one. Therefore, the number of values of θ for which one or more of the three directions is not in D is no greater than $3\cdot|S^{n-1}\backslash D|<2^{\aleph_0}$. There are $2^{\aleph_0}$ values of θ to choose from in this construction of a polygonal path from u to v. Therefore, there is a polygonal path of the required sort joining u and v.

Therefore, if $(k-1)r<d(u,v)<kr$, then $d(T(u),T(v))\leqq kr$ (whether or not $d(u,v)^{-1}(u-v)\in D$, in fact).

LEMMA 2.2. *If T preserves arbitrarily large and small distances in directions D, then T preserves all distances in direction D.*

The proof of Lemma 2.2, which depends heavily on Lemma 2.1, is almost identical to the proof of Lemma 2 in Bishop's paper. The necessary modifications, such as inserting the words "in directions D" in places, are obvious enough that we shall omit the proof here.

LEMMA 2.3. *If T preserves all distances in directions D, then T is an isometry.*

Proof. Suppose $u\neq v$, and $d(u,v)^{-1}(u-v)\notin D$. Let $\varepsilon>0$. Since D is a large set of directions, there is a point w such that $d(u,v)=d(u,w)$, $d(w,v)<\varepsilon$, $d(u,w)^{-1}(u-w)\in D$ and $d(v,w)^{-1}(v-w)\in D$. Then

$$\begin{aligned} d(T(u),T(v)) &\leqq d(T(u),T(w))+d(T(w),T(v))\\ &= d(u,w)+d(w,v)=d(u,v)+d(w,v)<d(u,v)+\varepsilon \end{aligned}$$

and

$$\begin{aligned} d(u,v)=d(u,w)=d(T(u),T(w)) &\leqq d(T(u),T(v))+d(T(v),T(w))\\ &= d(T(u),T(v))+d(v,w)<d(T(u),T(v))+\varepsilon. \end{aligned}$$

Thus $d(u,v)-\varepsilon<d(T(u),T(v))<d(u,v)+\varepsilon$. Since ε is arbitrary, $d(T(u),T(v))=d(u,v)$.

The two preceding lemmas provide the following scheme for the proof of Theorem 2: supposing $T\colon E^n\to E^n$ preserves the distance 1 in a large set of directions D, show that T preserves arbitrarily large and small distances in another large set of directions D'. For $n>2$, this plan may be executed by following, and appropriately modifying, the proofs of Bishop's Lemmas 3 and 4.

The case $n=2$ is special. We have not been able to modify Bishop's elegant treatment (for $D=S^1$) to produce a brisk proof of Theorem 2. What follows bears some resemblance, at essential points, to P. Zvengrowski's original proof for the plane, of the theorem that Bishop generalized to higher dimensions.

Proof of Theorem 2. First note that if T preserves the distance r in directions D, then the three vertices of an equilateral triangle of side length r get mapped by T into the vertices of an equilateral triangle of side length r, provided the sides of the original triangle represent directions in D.

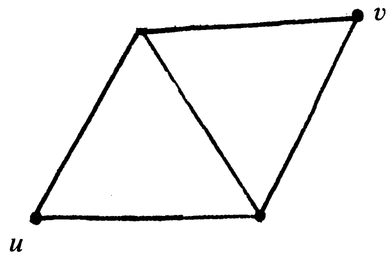

FIGURE 2

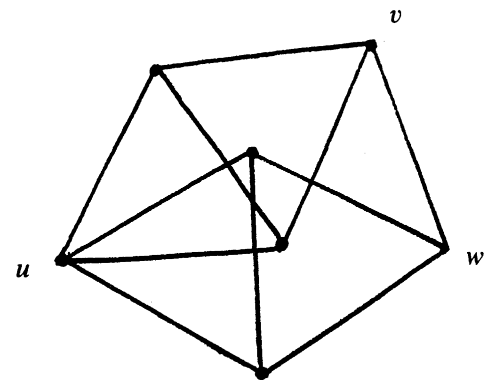

FIGURE 3

Now, suppose T preserves the distance $r > 0$ in directions D, and suppose $d(u, v) = r\sqrt{3}$. Then u and v are vertices in FIGURE 2, comprised of two equilateral triangles of side length r, with a common side. If the three directions represented by segments in FIGURE 2 are in D, then clearly either $T(u) = T(v)$, or $d(T(u), T(v)) = r\sqrt{3}$. There is a point w such that $d(w, v) = r$ and $d(w, u) = r\sqrt{3}$ (there are in fact two such points). If all seven of the directions represented by segments in FIGURE 3, including the segment joining v and w, are in D, then it must be the case that $d(T(u), T(v)) = r\sqrt{3}$. For if $T(u) = T(v)$ then either $T(w) = T(v)$ or $d(T(v), T(w)) = r\sqrt{3}$ and either of these would contradict the assumption that T preserves the distance r in directions D.

Thus, if $d(u, v) = r\sqrt{3}$, and every one of seven directions, each uniquely associated with the direction of the segment joining u and v, lies in D, then $d(T(w), T(v)) = r\sqrt{3}$. It follows that T preserves the distance $r\sqrt{3}$ in a large set of directions.

So if T preserves the distance 1 in a large set of directions, then T preserves the distance $(\sqrt{3})^k$ in a large set of directions, for each positive integer k. Taking the intersection of this countable collection of large sets of directions, we obtain a large set of directions, which we continue to call D, such that T preserves all the distances $(\sqrt{3})^k$, $k = 0, 1, 2, \ldots$ in directions D.

Now, suppose u_0, u_1 and u_2 are points spaced one apart,and in that order, on a line. Let v and w be points such that u_0, v, w, u_2 is half of a regular hexagon of side length 1 (see FIGURE 4).

Note that $d(v, u_2) = \sqrt{3} = (u_0, w)$. If all five of the directions represented by the nine segments in FIGURE 4 are in D, then T must respect this figure by sending u_0, u_1, u_2, v and w into points playing the same respective roles in a congruent figure.

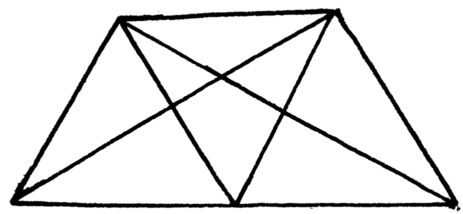

FIGURE 4

It follows not only that $d(T(u_0), T(u_2))=2$ but also that $T(u_0)$, $T(u_1)$ and $T(u_2)$ are collinear, spaced one apart on the line they determine, and in the "right" order. Since this conclusion about $T(u_0)$, $T(u_1)$ and $T(u_2)$ only requires inclusion in D of five directions, each uniquely associated with the direction of the segment $u_0u_1u_2$, it follows that T will preserve collinearity, order, and spacing of three points spaced one apart on a line pointing in any one of a large set of directions, which we continue to call D.

Now we claim that T will preserve the collinearity, order, and spacing of k points, spaced one apart, on any line pointing in one of these same directions D, for any positive integer k. Suppose this is known to be true for some $k \geqq 3$. Given $k+1$ points, $u_0, u_1, \ldots, u_k$ spaced one apart, in the order of their indices, on a line pointing in a direction in D: note that $T(u_k)$ must lie on the intersection of the unit circle centered at $T(u_{k-1})$ and the circle of radius $k-1$ centered at $T(x_1)$, by application of the induction hypothesis. By induction, then, the claim of this paragraph is proven.

Similarly, since T preserves the distance $\sqrt{3}$ in a large set of directions, it follows that there is a large set of directions D' such that T preserves the collinearity, order, and spacing of points spaced $\sqrt{3}$ apart along a line pointing in one of the directions D'.

We intersect D' and D to obtain another large set of directions, which we continue to call D.

Suppose L is a line pointing in one of the directions of D, and, for simplicity, consider L to be a number line, with 0 and 1 distinguished. Now, the points $T(n)$, $n=0, \pm 1, \pm 2, \ldots$ are collinear, spaced one apart and in the right order. Also, the points $T(n\sqrt{3})$, $n=0, \pm 1, \pm 2, \ldots$ are collinear, spaced $\sqrt{3}$ apart, and in the right order. We would like to show that the points $T(n\sqrt{3})$ are collinear with the points $T(n)$, and that all of these points wind up in the right order. It suffices to show that $T(\sqrt{3})$ lies on the same line as the $T(n)$, and in the right place on that line. For any $\varepsilon>0$, there are positive integers n and m such that

$$n-m\sqrt{3}<\sqrt{3}<n-m\sqrt{3}+\varepsilon.$$

If $\varepsilon>0$, and n and m are as above, we have

$$m\sqrt{3}-\varepsilon<n-\sqrt{3}<m\sqrt{3}.$$

For ε sufficiently small, so that both $\varepsilon<\sqrt{3}$ and $m \geqq 3$, it follows from Lemma 2.1 that

$$d(T(\sqrt{3}), T(n)) \leqq m\sqrt{3}.$$

We also know that $d(T(\sqrt{3}), T(0))=\sqrt{3}$, and $d(T(0, T(n))=n$. Thus, $T(\sqrt{3})$ lies on or inside the circle of radius $m\sqrt{3}$ about $T(n)$, and on the circle of radius $\sqrt{3}$ about $T(0)$, while $T(n)$ is a distance n from $T(0)$. Note also that

$$0<m\sqrt{3}-(n-\sqrt{3})<\varepsilon.$$

It follows that $T(\sqrt{3})$ lies on the arc A of Figure 5, and that the segment s of that figure, being of length $m\sqrt{3}-(n-\sqrt{3})$, is of length less than ε.

As we let ε go to zero the length of arc A is forced to zero and thus $T(\sqrt{3})$ is collinear with $T(0)$, $T(1)$, etc., and in the right place on the line containing them, and therefore so are $T(n\sqrt{3})$, $n=0, \pm 1, \pm 2, \ldots$.

Now, L was any line pointing in a direction in D; as an easy consequence of the foregoing, we have that T must preserve all positive distances of the form $n-m\sqrt{3}$, in directions D, with n, m positive integers; for, given u and v, with $d(u, v)=n-m\sqrt{3}$ and $d(u, v)^{-1}(u-v) \in D$, take L to be the line determined by u and v, and let w be the point on L a distance $m\sqrt{3}$ from u and a distance n from v. Think of w as being 0, u as being $m\sqrt{3}$, v as being n, on the number line L. Then what has

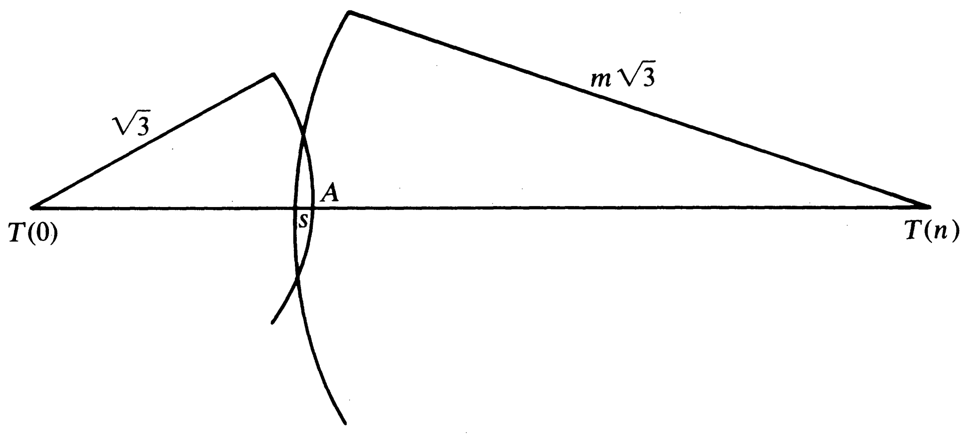

FIGURE 5

been shown about the respect that T has for collinearity, order, and spacing of certain points on L implies that $d(T(u), T(v)) = n - m\sqrt{3}$.

Thus T preserves arbitrarily small, as well as arbitrarily large, distances in directions D, which finishes the proof of the theorem, in view of Lemmas 2.2 and 2.3.

The following corollary is not hard to prove directly, but comes very easily from Theorems 1 and 2.

COROLLARY 2.4. *If* $|S^{n-1}\backslash D| < 2^{\aleph_0}$, $n \geqq 2$, *then* $A(D) = E^n$.

4. Functions from E^n to E^m

Suppose $T: E^n \to E^m$, $2 \leqq n < m$ preserves the distance 1. Does T necessarily preserve all distances? We have no answer. However, a modification of an example of Beckman and Quarles [1] provides some information pertinent to the question.

Let E^∞ denote the real or complex Hilbert space of square summable sequences with the usual metric produced by the usual norm $\|\cdot\|$, defined by $\|(a_i)\| = (\sum_{i=1}^{\infty} |a_i|^2)^{1/2}$. For each $n = 1, 2, \ldots$, let e_n denote the sequence with 1 in the nth place, zeros elsewhere.

PROPOSITION 3. *If* (X, d) *is a metric space possessing a proper countable dense subset* $Y = \{y_n : n = 1, 2, \ldots\}$, *then there is a function* T *from* X *to* E^∞ *which preserves the distance* 1 *but does not preserve all distances*; T *is not, in fact, one-to-one.*

Proof. Clearly there is a function $S: X \to Y$ such that $S(y) = y$ for each $y \in Y$ and $d(x, S(x)) < 1/2$ for each $x \in X$. Now define $R: Y \to E^\infty$ by $R(y_n) = (1/\sqrt{2})e_n$, and set $T = R \circ S$. If $w, x \in X$ and $d(w, x) = 1$, then $S(w) = S(x)$, whence $\|R \circ S(w) - R \circ S(x)\| = 1$, so T preserves the distance 1. On the other hand, since Y is a proper subset of X, there is some $x \in X \backslash Y$; then $x = S(x)$, but $T(x) = T \circ S(x)$, so T is not one-to-one.

The hypothesis that Y is a proper subset of X cannot be deleted in Proposition 3; to see this, take X to be a countable set, and d the trivial metric satisfying $d(x, y) = 1$ if $x \neq y$.

Obviously Proposition 3 should severely modify our hopes for the truth of statements that are like the result of Beckman and Quarles, but in which the domain and range of T are varied.

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Permutation Numbers

Characteristics for relations between certain numbers and their products.

JEFFREY JAFFE, student
Massachusetts Institute of Technology

The study of natural numbers is a vast field and over the years has encompassed many areas. One of these relates to factorization and divisibility, where an investigator determines how a number can be broken down into smaller parts or factors. There are many results of various degrees of complexity related to this topic, for example, the well-known facts that a number is divisible by nine if and only if the sum of its digits is divisible by nine, or that a number is divisible by eleven if and only if the alternating sum and difference of its digits is divisible by eleven.

Interestingly enough, the reverse process has not obtained much attention. By the reverse process I mean the process of determining the relationship between a number and the set of products of it with other numbers. For instance, we might wonder why $0123456789 \times 13 = 1604938257$ has all ten digits in both the multiplicand and the product, and when we can predict such a result. Here too, we might expect the sum of the digits of the number, or some other simply identifiable feature, to provide a key to determine when this characteristic exists. In this paper we establish such a simple feature. While this result is quite interesting in its own right, its main value probably lies in its suggestion for future research in other phenomena relating to products (rather than factors) of numbers.

In order to introduce our principal theorem, we will need one simple definition: A **permutation number** of base b is a b digit number in the base b where every digit $0, 1, \ldots, b-1$ appears exactly once. (For example, 2340678195_{10} and 03214_5 are permutation numbers.) The result that we will discuss refers to the first permutation number A of the base b (i.e., $A = 0123\ldots b-1_b$), in particular, to a certain curious phenomenon best illustrated in the familiar case where $b = 10$. The product $0123456789 \times n$ yields a different permutation number for 33 different choices of n. This is remarkable in light of the fact that for $n > 81$ the product is no longer even a ten digit number, one of the requirements for a number to be a permutation number. Our principal theorem explains this phenomenon:

THEOREM. *The product An is a permutation number if and only if*

(1) $(n, b-1) = 1$ *and*

(2) *the sum of the digits of n is less then* $b-1$,

where $(n, b-1)$ *is the greatest common divisor of n and* $b-1$.

For example, in base 10, $2A = 024691357$ is a permutation number since $(2,9) = 1$ and $2 < 9$; $3A = 0370370367$ is not a permutation number since although $3 < 9$, $(3,9) \neq 1$; and $19A = 2345678991$ is also not a permutation number since although $(19,9) = 1$, $10 \not< 9$. Following our (lengthy) proof of this theorem we will deduce as an immediate corollary that the number of times that the product nA is a permutation number in the base b, is $\frac{1}{2}(b+1)\phi(b-1)$, where ϕ is Euler's ϕ function.

The proof of the theorem will be broken down into three cases. Throughout the discussion we will give examples where $b = 10$ to help clarify the ideas. The first case to be proved will be: if $(n, b-1) \neq 1$ then An is not a permutation number. This case consists of a simple straightforward calculation. The second case consists of a detailed study of $n < b-1$ (where, quite clearly the sum of the digits of n is less than $b-1$), and we determine that if in addition $(n, b-1) = 1$, then no digit in the b digit product repeats itself. Inasmuch as, in this case, we set up quite extensive machinery to prove this (machinery which will also be used for the third case), the calculations become quite tedious. Therefore we will omit some of the less enlightening remarks.

Finally, if $n > b-1$, we reduce the problem to the case of $n < b-1$, provided the sum of the digits of n is less than $b-1$. We demonstrate how a formula can be obtained for each digit of the large number, based on the sum of its digits. For instance, we derive directly that 43_{10} yields a permutation number from the fact that 7, the sum of its digits, is less than 9 and also yields a permutation number.

PROPOSITION 1. *If* $(b-1, n) \neq 1$ *then An is not a permutation number.*

Proof. First note that $A(b-1) = 11\ldots101_b$ for all b (e.g., $0123456789 \times 9 = 1111111101$). Let $(n, b-1) = p$. Since p divides both n and $b-1$, there are numbers m and r such that (1) $n = mp$ and (2) $b-1 = rp$. From (2), $A(b-1) = Arp$; from (1), $p = n/m$; so $A(b-1) = Arn/m = 11\ldots101_b$ as above. Multiplying both sides by m/r, we get $An = (m/r)(11\ldots101_b)$.

Now that we have obtained an expression for An, we wish to show that a digit repeats itself; specifically that the first digit (from the left) is the same as the $(r+1)$st digit. Equivalently we will show that r divides $y = mb^{b-1} + \cdots + mb^{b-r}$. This would result in both the first and $(r+1)$st digits being the greatest integer in m/r (i.e., the same one!).

To show that r divides y, let $z = b^{b-r}$. Then $y = mz(1 + \cdots + b^{r-1})$. Consider $y - mzr$. Note that

$$y - mzr = mz(1 - 1 + \cdots + b^{r-2} - 1 + b^{r-1} - 1).$$

Since r divides $b-1$, r divides $b^i - 1$ for all i, and thus we have that r divides $y - mzr$. Finally, since r divides mzr, r divides y as desired.

PROPOSITION 2. *If* $(n, b-1) = 1$ *and* $n < b-1$ *then An is a permutation number.*

Proof. The proof of this proposition is most easily done in three parts. The first part is a lemma which inductively derives an expression for a position in the product An, given the position to its right (e.g., for $2A = 0246913578$, one could derive the fact that a 5 comes next after the 7 and 8). The second part of the proof uses this lemma to demonstrate how one may derive a similar inductive expression, this time working from left to right. The third part of the proof then breaks the digits $0, \ldots, b-1$ into equivalence classes, and demonstrates that no equivalence class of digits repeats itself in the product. Since the three parts of the proof are similar and a bit repetitious, we will omit some of the calculation.

LEMMA. *Let* c_i *be the number carried into the ith position (from the left) from the* $(i+1)$*st position in the usual process of multiplying A by n where* $0 < n < b-1$. *Also, let* t_i *be the digit appearing in the ith position of the product. Then:*

(1) $c_b = 0$
(2) $t_b = b - n$
(3) $c_{b-1} = n - 1$
(4) $t_i = t_{i+1} - n - c_{i+1} + b(c_i - c_{i-1}) + c_i$
(5) $c_i = c_{i+1}$ *if* $t_{i+1} \leqq b - n - 1$ *for* $i < b - 1$
$c_i = c_{i+1} - 1$ *if* $t_{i+1} > b - n - 1$ *for* $i < b - 1$

Proof. (1) is obvious. To prove (2) and (3), we look at the last few digits, obtaining

$$(b-1)(n) = nb - n = (n-1)b + (b-n).$$

So $b - n$ appears in the bth place and $n - 1$ is carried into the $(b-1)$st place. (4) follows by definition of multiplication: $(i-1)n + c_i = bc_{i-1} + t_i$, and similarly, $in + c_{i+1} = bc_i + t_{i+1}$. Subtracting terms we get the desired result.

To prove (5), we use induction on i (working backwards from $b-2$).

Basis step: $((b-2)b + b - 1)n = (b^2 - b - 1)n = (n-1)b^2 + (b-n-1)b + (b-n)$. Therefore $c_{b-2} = c_{b-1} = n - 1$ as desired since $t_{b-1} = b - n - 1 \leqq b - n - 1$.

Inductive step: To complete the induction we need to show that the process of multiplication guarantees that the correct number is carried over in both cases (i.e., $t_i \leqq b - n - 1$ and $t_i > b - n - 1$). The proofs for the two cases are similar calculations so for simplicity we will prove only the case where $t_i > b - n - 1$.

Assume then (inductively), that the entire lemma is true for $j = b-2, \ldots, b-i$. We want to show for $j = b - i - 1$, that if $t_{b-i} > b - n - 1$ then $c_{b-i-1} = c_{b-i} - 1$. From part (4) of the lemma, we have

$$t_{b-i+1} = t_{b-i} + n + c_{b-i+1} - c_{b-i} - b(c_{b-i} - c_{b-i-1}).$$

Since $b - 1 > t_{b-i+1}$ and by induction $c_{b-i+1} \geqq c_{b-i}$, we get by transposing

$$b - n - 1 \geqq t_{b-i} - b(c_{b-i} - c_{b-i-1}).$$

Since $t_{b-i} > b - n - 1$, we get that $0 > b - n - 1 - t_{b-i} \geqq -bc_{b-i} + bc_{b-i-1}$ which yields $c_{b-i} > c_{b-i-1}$. Now all we need to show is that the difference between c_{b-i} and c_{b-i-1} is in fact no more than one.

To this end, assume that $c_{b-i} - c_{b-i-1}$ is indeed greater than or equal to two. Then from part (4) of the lemma:

$$t_{b-i+1} = n + c_{b-i+1} - c_{b-i} - b(c_{b-i} - c_{b-i-1}) + t_{b-i}.$$

Since $t_{b-i+1} \geqq 0$, if we assume $c_{b-i} - c_{b-i-1} \geqq 2$ we obtain

$$2b \leqq n + c_{b-i+1} - c_{b-i} + t_{b-i}.$$

Since $t_{b-i} \leqq b - 1$, and by induction $c_{b-i+1} - c_{b-i} \leqq 1$, we obtain

$$2b \leqq n + 1 + b - 1 \qquad \text{or} \qquad b \leqq n$$

contradicting the fact that $n < b - 1$.

While this characterization provides us with a method of moving from right to left in the product, what we will in fact need is the ability to move from left to right in order to show that each digit $0, \ldots, b-1$ appears exactly once in the product. What results from the above lemma, as the unique possibility for moving from left to right in the product (for $i < b - 2$) is:

RULE 1. If $t_i + n < b - n - 1$ then $t_{i+1} = t_i + n$.

RULE 2. If $t_i + n \geqq b$ then $t_{i+1} = t_i + n - b$ unless $t_i + n - b > b - n - 1$ in which case $t_{i+1} = t_i + n + 1 - b$.

RULE 3. If $b > t_i + n > b - n - 1$ then $t_{i+1} = t_i + n + 1$.

RULE 4. $t_{b-1} = b - n - 1$; $t_b = b - n$ (by direct calculation).

What we will now do is break the digits $0, \ldots, b-1$ into n equivalence classes, and demonstrate that each equivalence class in its entirety appears only once in the product. For clarity, it is convenient to use an example, and we will use $4A_{10} = 0493827156$. In our example we will show that each of the four groupings $(\{0,4,9\}; \{3,8\}; \{2,7\}; \{1,5,6\})$ must appear in the product. We number the equivalence classes as follows:

$$\begin{aligned}
&\text{Class } n-1 = \{b-1, b-n-2, b-2n-2, b-3n-2, \ldots\}\\
&\quad\vdots\\
&\text{Class } n-i = \{b-i, b-n-i-1, b-2n-i-1, \ldots\}\\
&\quad\vdots\\
&\text{Class } 0 \quad\;\; = \{b-n, b-n-1, b-2n-1, \ldots\}
\end{aligned}$$

with class 0 taking on a different form since it corresponds to the digits occurring on the far right of the product. Thus in our example, we have ($n = 4$): Class $3 = \{9,4,0\}$; class $2 = \{8,3\}$; class $1 = \{7,2\}$; class $0 = \{6,5,1\}$.

Rules 1 and 3 force the next digit to the right (after any particular digit) to be the next greatest number in the equivalence class, provided we haven't already reached the maximum for the class. Rule 2 acts on the largest number of each class, and forces the next number to be the smallest number in a different class. In particular, it forces the $(i-b+n+1)$st class to follow the ith class (mod n). Since $b-n-1$ is relatively prime to n, successive subtractions of $b-n-1$ from the various class numbers causes an encounter with each class. Thus each class is encountered once by Rule 2, and each element in it is used by Rules 1 and 3.

PROPOSITION 3. *If* $(n, b-1) = 1$ *and* $n > b-1$ *then* An *is a permutation number iff the sum of the digits of* n *is less than* $b-1$.

Proof. Write n as $n = q(b-1) + m$ where $m < b-1$, $(m, b-1) = 1$ (e.g., $22 = 2(9)+4$). We then can write nA as $Aq(b-1) + Am$.

Case 1. $q < m$.

$$A(b-1) = 11\ldots101_b \text{ so therefore } Aq(b-1) = qq\ldots q0q_b.$$

Therefore, each digit of An is exactly q greater than the corresponding digit of Am except for carries and the $b-1$st position. Specifically, we can show that each digit $0, \ldots, b-1$ appears in the product An if $q = 1, \ldots, m-1$ as follows:

(1) For a digit $d = 0, \ldots, \min(q-1, b-m-2)$ we have that d appears in the same position that $b+d-q$ appeared in the product for Am.

(2) If $b-m-2 > q$ then a digit $d = q, \ldots, b-m-2$ appears in the same position that $d-q$ appeared in for Am.

(3) For a digit $d = b-m-1$, d appears in the $b-1$st position.

(4) If $q > b-m$ then for a digit $d = b-m, \ldots, q$, we have that d appears in the same position as $b+d-q-1$ appeared in for Am.

(5) For a digit $d = \max(b-m, q+1), \ldots, b-m+q-1$, we have that d appears in the same position as $d-q-1$ appeared in for Am.

(6) Finally for a digit $d = b-m+q, \ldots, b-1$, we have that d appears in the same position as $d-q$ appeared in for Am.

The proofs for the above six assertions are similar, so we will be content to prove just the first. Assume $0 < d \leqq \min(q-1, b-m-2)$, and that $b+d-q$ appeared in the ith position of the product Am. Since $An = Aq(b-1) + Am$, we must add q to the ith position to see what results for An.

Adding q to $b+d-q$ results in $b+d$ which is greater than b, so in effect we get d in the ith position with one carried into the $(i-1)$st position. All we must show then, is that nothing was carried from the $(i+1)$st position into the ith position (for then we would be getting a total of $d+1$ (not d) for this position of the product, contrary to our assertion).

To show that, we will return to the rules derived earlier to determine t_{i+1} in the product Am (denoted $t_{i+1,m}$). Since $m > q$ we have that $t_{i,m} + m \geqq b$ and therefore Rule 2 applies in determining $t_{i+1,m}$. Therefore $t_{i+1,m} \leqq t_{i,m} + m - b + 1 = b + d - q + m - b + 1$ or $t_{i+1,m} \leqq d + m - q + 1$. To get $t_{i+1,n}$, we just add q and we see $t_{i+1,n} \leqq d + m - q + 1 + q = d + m + 1$. Since $d \leqq b - m - 2$, we can rewrite the above as $t_{i+1,n} \leqq b - m - 2 + m + 1 = b - 1$ which means that no carry was necessary into the ith place. (Note that for the $(i+1)$st position we did not worry about adding $q+1$ because of a carry from the $(i+2)$nd position. The reason for this is that adding $q+1$ would have produced a carry into the ith position only if $d = b - m - 2$, but in that case $t_{i+2,m}$ would have been too low to have the $(i+2)$nd position merit to have a carry into the $(i+1)$st position.)

Case 2. $q = m$.

If $q = m$, then $An = A((b-1)q + m) = ((b-1)m + m)A = bmA$. That is to say that An has the identical form as Am except that the whole number is multiplied by b, resulting in a zero in the rightmost place (instead of the leftmost one) and all the digits shifting left one place. Thus clearly, here too, An is a permutation number. This is in agreement with the proposition since $q \leqq m$ corresponds to those times that the sum of the digits is less than $b-1$.

Case 3. $q > m$.

For $q = m+1$, $b-n$ appears both where $b-n-1$ appeared for $q = m$, and where $b-n$ appeared. Similarly for any $q > m$ we get repeating digits. Since $q > m$ in our formulation occurs iff the sum of the digits of n exceeds $b-1$, the proposition is proved.

To summarize the theorem, Proposition 1 tells us that a necessary condition for An to be a permutation number is $(n, b-1) = 1$. Propositions 2 and 3 indicate that this condition coupled with the condition that the sum of the digits of n be less then $b-1$ is indeed necessary and sufficient. We now repeat and prove the corollary to this theorem:

COROLLARY. *The number of numbers n such that An is a permutation number is* $\frac{1}{2}(b+1)\phi(b-1)$.

Proof. The $\phi(b-1)$ numbers less than and relatively prime to $b-1$ all produce permutation numbers when multiplied by A. By the construction used in the proof of Proposition 3, we see that if $n < b-1$ and $(n, b-1) = 1$, there are $n+1$ numbers m_i such that Am_i is a permutation number and the sum of the digits of m_i equals n. If we pair n and $b-n-1$ (as we may do since $(b-1, n) = 1$ implies $(b-1, b-n-1) = 1$) we see that the average number of permutation numbers with sum of digits equal to any given n relatively prime to $b-1$ is $(n+1+b-n)/2 = (b+1)/2$. Hence the total number of numbers that produce permutation numbers is $\frac{1}{2}(b+1)\phi(b-1)$.

Square them

Round numbers are always false.

— SAMUEL JOHNSON

NOTES

Hex Must Have a Winner: An Inductive Proof

DAVID BERMAN
University of New Orleans

The game of Hex is an excellent example of a game for which a winning strategy is known to exist, even though it is not known what the strategy is. It is easy to show the existence of a strategy once it is known that either black or white must win, that is, that Hex cannot end in a draw. The available proofs of the latter fact are all rather difficult (see, for instance, [**1**, pp. 334–338]). In this note we give a simple proof.

A Hex board is a parallelogram divided into m rows and n columns of hexagons. Players alternate turns placing black and white stones on the board with the objective of completing a chain from one side of the board to the opposite side, one player seeking a chain joining top to bottom, the other joining right to left. We will show that the game cannot end without a winner. Specifically, whenever a Hex board is completely filled with black and white stones, there must be either a black chain from right to left or a white chain from top to bottom. And, equivalently, there must be either a white chain from right to left or a black chain from top to bottom. Consequently, one of the players must have achieved his objective and won the game.

We will prove our proposition by induction on m and n, the dimensions of the board. The proposition is clear for a $1 \times n$, $m \times 1$, or 2×2 board. Now we assume it true for any board smaller than $m \times n$. Consider the $(m-1) \times n$ board obtained by deleting row m. By the inductive hypothesis there is either a black chain from column 1 to column n or else a white chain W_1 from row 1 to row $m-1$. In the former case we are done, so we assume the latter. It follows from an analogous argument involving deletion of row 1 that there must be a white chain W_2 from row 2 to row m. We assume that W_1 and W_2 do not meet, or else we would be done.

By deleting column n and then column 1 we can show in a similar manner that there are nonintersecting black chains B_1 from column 1 to column $n-1$ and B_2 from column 2 to column n. Since these horizontal black chains do not meet, the number of rows m must be greater than 2.

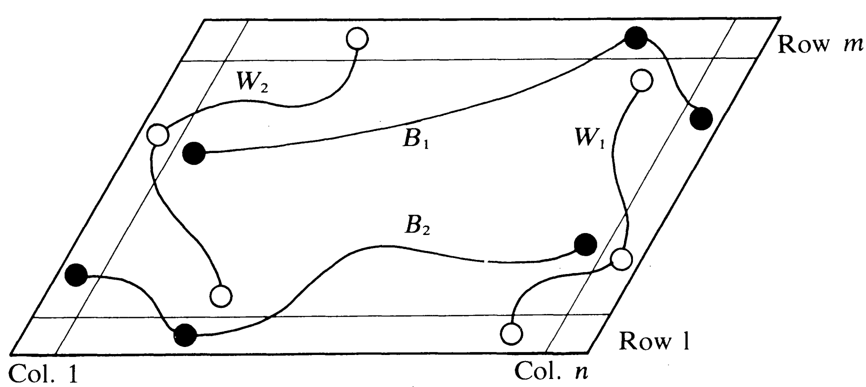

FIGURE 1

Similarly, since the vertical chains W_1 and W_2 do not meet, the number of columns n must be greater than 2.

We now consider the $(m-2)\times(n-2)$ board (See FIGURE 1) obtained by deleting rows 1 and m and columns 1 and n. In this board we apply the inductive hypothesis in its equivalent form: there must be either a white chain from column 2 to column $n-1$ or a black chain from row 2 to row $m-1$. We assume, without loss of generality, the former. This chain W_3 must intersect chains W_1 and W_2, so W_1, W_2 and W_3 together form a white chain from row 1 to row m. This completes the proof.

We note, in conclusion, that this proof can easily be modified to deal with other games of this sort, for instance, Bridge-it.

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A Double Butterfly Theorem

DIXON JONES
College, Alaska

To the extensive annals of geometric lepidopterology we add a further modification of the well-known butterfly problem. Let us define a "**butterfly,**" denoted by $)B($, as the two triangles formed by the diagonals and two opposite sides of a convex quadrilateral, and refer to these triangles as

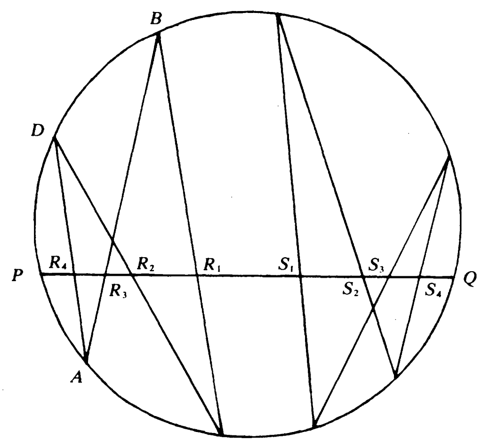

FIGURE 1.

"wings" of the butterfly. FIGURE 1, depicting two butterflies inscribed in a circle, illustrates our main result:

THEOREM. *Let PQ be a fixed chord of a circle. Let $)R($ and $)S($ be inscribed in the circle and oriented such that their wings cut PQ (in order from left to right) at R_4, R_3, R_2, R_1, and S_1, S_2, S_3, S_4, respectively. If $PR_1 = QS_1$, $PR_2 = QS_2$, and $PR_3 = QS_3$, then $PR_4 = QS_4$.*

Proof. Consider $)R($. Denoting by $(UVWX)$ the double ratio on points U, V, W, and X, we have

$$(PR_4R_3Q) = \frac{\sin\angle PAB}{\sin\angle BAD} \div \frac{\sin\angle PAQ}{\sin\angle QAD}, \qquad (PR_2R_1Q) = \frac{\sin\angle PCB}{\sin\angle BCD} \div \frac{\sin\angle PAQ}{\sin\angle QCD}.$$

Since angles inscribed in the same arc are equal, we deduce that $(PR_4R_3Q) = (PR_2R_1Q)$. Expanding and cancelling, we have

$$(1) \qquad \frac{PR_3 \cdot QR_4}{R_3R_4} = \frac{PR_1 \cdot QR_2}{R_1R_2}.$$

Identical reasoning for $)S($ establishes that $(QS_4S_3P) = (QS_2S_1P)$, by which we obtain

$$(2) \qquad \frac{QS_3 \cdot PS_4}{S_3S_4} = \frac{QS_1 \cdot PS_2}{S_1S_2}.$$

Since in the given conditions $QS_1 = PR_1$, $PS_2 = QR_2$, and $S_1S_2 = R_1R_2$, substitution in (2) gives us $QS_3 \cdot PS_4 : S_3S_4 = PR_1 \cdot QR_2 : R_1R_2$ which with (1) yields $QS_3 \cdot PS_4 : S_3S_4 = PR_3 \cdot QR_4 : R_3R_4$ or, since $QS_3 = PR_3$,

$$(3) \qquad PS_4 \cdot R_3R_4 = QR_4 \cdot S_3S_4.$$

Now, $PS_4 = PS_3 + S_3S_4$ and $QR_4 = QR_3 + R_3R_4$. Substituting these into (3) yields

$$PS_3 \cdot R_3R_4 + S_3S_4 \cdot R_3R_4 = QR_3 \cdot S_3S_4 + R_3R_4 \cdot S_3S_4.$$

Cancelling the identical terms and taking into account the fact, evident from the given conditions, that $PS_3 = QR_3$, we have $R_3R_4 = S_3S_4$ which by subtraction gives us the result $PR_4 = QS_4$.

The reader may verify various special cases of the theorem which occur when, in addition to the given conditions, (a) $R_3 = R_2$ and $S_3 = S_2$, (b) $R_1 = S_1$, (c) $R_3 = R_2 = S_1$ and $S_3 = S_2 = R_1$, (d) all intersection points are the same, which is a reduction to the original butterfly problem, or (e) $R_3 = R_2 = R_1 = S_1 = S_2 = S_3$, which produces Klamkin's extension of the problem [**1**].

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The Arithmetic Mean–Geometric Mean Inequality: A New Proof

KONG-MING CHONG

University of Malaya, Kuala Lumpur

In this note, we give a simple inductive proof for the arithmetic mean–geometric mean inequality. Other inductive proofs already known can be found in [**1**, §5, pp. 4–5, §11, pp. 9–10, and §13, pp. 11–12], [**2**, p. 46], [**3**, §2.6, pp. 18–21], [**4**] and [**5**, pp. 285–286].

The inequality concerned is

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n} \geqq \sqrt[n]{a_1a_2\cdots a_n} = G,$$

where $a_1, a_2, \ldots, a_n$ are positive numbers. There is equality if and only if $a_1 = a_2 = \cdots = a_n$.

Suppose that $a_1 \geqq a_2 \geqq \cdots \geqq a_n$. Then, clearly, $a_1 \geqq G \geqq a_n$, with equality if and only if $a_1 = a_2 = \cdots = a_n$, and so

$$(1) \qquad a_1 + a_n - \left(G + \frac{a_1a_n}{G}\right) = \frac{1}{G}(a_1 - G)(G - a_n) \geqq 0.$$

Now, the result is true for $n = 2$. Suppose that it is true for $n - 1$. Since the geometric mean of the numbers $a_2, a_3, \ldots a_{n-1}$ and $a_1 a_n / G$ is G, the induction hypothesis implies that

$$G \leqq \frac{a_2 + a_3 + \cdots + a_{n-1} + \dfrac{a_1 a_n}{G}}{n-1}$$

i.e.,

$$nG \leqq a_2 + a_3 + \cdots + a_{n-1} + \left(G + \frac{a_1 a_n}{G}\right)$$

or

$$nG \leqq a_1 + a_2 + \cdots + a_n$$

by virtue of (1).

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A Generalized Parallelogram Law

ALI R. AMIR-MOEZ

J. D. HAMILTON

Texas Tech University

Pythagoras' famous theorem about right triangles can be rephrased by saying that the sum of the squares of the lengths of the two diagonals of a rectangle is equal to the sum of the squares of the lengths of the four sides. This result was shown by Apollonius in ancient times to hold for any parallelogram. If one thinks in terms of two vectors ξ, η in Euclidean n-space, E^n, Apollonius' theorem may, because of the parallelogram law of vector addition, be stated as $\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2\|\xi\|^2 + 2\|\eta\|^2$, where $\| \quad \|$ denotes the usual norm defined in terms of the inner product: $\|\xi\|^2 = (\xi, \xi)$. Now in 1935 [**1**] Jordan and Von Neumann showed that any norm which satisfies the equality in Apollonius' theorem is in fact a norm derived from an inner product, for the equation $(\xi, \eta) = \frac{1}{4}[\|\xi + \eta\|^2 - \|\xi - \eta\|^2]$ defines a valid inner product such that $\|\xi\| = (\xi, \xi)^{\frac{1}{2}}$.

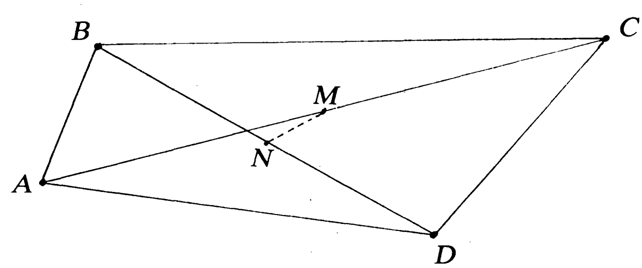

FIGURE 1

We present here a generalization of Apollonius' theorem which makes an interesting geometric application of vector algebra. *Let ABCD be any quadrilateral. If M and N are the midpoints of the diagonals AC and BD, respectively, then* $(AB)^2+(BC)^2+(CD)^2+(DA)^2=(AC)^2+(BD)^2+4(MN)^2$ (*see* FIGURE 1). Basically $4(MN)^2$ is the appropriate correction factor in the case that the two diagonals do not bisect each other. To see this we use a coordinate system with A at the origin and think of each of the points B, C, D, M, N as a vector β, γ, δ, μ, ν respectively. Now $\mu=\frac{1}{2}\gamma$ and $\nu=\frac{1}{2}(\beta+\delta)$, so by direct computation

$$4(MN)^2=\|\beta+\delta-\gamma\|^2=\|\beta\|^2+\|\delta\|^2+\|\beta-\gamma\|^2+\|\delta-\gamma\|^2-\|\beta-\delta\|^2-\|\gamma\|^2$$

which is our original statement in vector language.

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The Quadratic Character of 2 mod p

KENNETH S. WILLIAMS

Carleton University

Here is a very simple proof of the result that

$$\left(\frac{2}{p}\right)=(-1)^{(p^2-1)/8}$$

for p an odd prime, where the Legendre symbol $(2|p)$ is $+1$ if 2 is a perfect square mod p and -1 otherwise. In other words, 2 is a perfect square mod p if and only if $(p^2-1)/8$ is even, i.e., if and only if $p\equiv 1$ or 7 (mod 8). The idea is to look at the number N_p of ordered pairs (x,y) — incongruent modulo p — which satisfy

(1) $$x^2+y^2\equiv 4 \pmod p,\quad x\not\equiv 0 \pmod p \quad\text{and}\quad y\not\equiv 0 \pmod p.$$

As the number of z with $z^2\equiv 2 \pmod p$ is given by $1+(2|p)$, the number of solutions (x,y) of (1) with $x\equiv \pm y \pmod p$ is $2(1+(2|p))$. Now each solution (x,y) with $x\not\equiv y \pmod p$ (if any) gives rise to eight distinct solutions of (1), namely $(\pm x,\pm y)$, $(\pm y,\pm x)$, so that we have

(2) $$N_p\equiv 2+2\left(\frac{2}{p}\right) \pmod 8.$$

Next, transforming the variables x, y to y, t by means of the transformation $x\equiv(2-y)t \pmod p$ we see that all the solutions of (1) are given by

$$(x,y)\equiv\left(\frac{4t}{t^2+1},\frac{2(t^2-1)}{t^2+1}\right) \pmod p$$

with $2\leqq t\leqq p-2$, $t^2\not\equiv -1 \pmod p$. Thus we have

(3) $$N_p=p-3-\left\{1+\left(\frac{-1}{p}\right)\right\}=p-4-(-1)^{(p-1)/2}.$$

Putting (2) and (3) together we obtain

$$\left(\frac{2}{p}\right) \equiv \tfrac{1}{2}(p-(-1)^{(p-1)/2})-3 \pmod 4$$

$$\equiv \begin{cases} +1 \pmod 4, \text{ if } p \equiv 1,7 \pmod 8, \\ -1 \pmod 4, \text{ if } p \equiv 3,5 \pmod 8. \end{cases}$$

As $\left(\frac{2}{p}\right) = \pm 1$ and $\frac{p^2-1}{8} \equiv \begin{cases} 0 \pmod 2 \Leftrightarrow p \equiv 1,7 \pmod 8, \\ 1 \pmod 2 \Leftrightarrow p \equiv 3,5 \pmod 8, \end{cases}$

the required result follows.

Spaces in which Compact Sets are Closed

JAMES E. JOSEPH
The Federal City College

It is known that if the graph $G(g)$ of a function $g: X \to Y$ is compact and compact subsets of X are closed (compact subsets of Y are closed), then g is continuous (closed) ([**1**], [**2**]). In this note, we prove the following:

THEOREM. *If X is a compact space, the following statements are equivalent:*
(1) *Compact subsets of X are closed.*
(2) *Any function with a compact graph from X to a space is continuous.*
(3) *Any function with a compact graph from a space to X is closed.*

Proof. In what follows, let π_x and π_y be the projections from $X \times Y$ onto X and Y respectively. To show that (1) implies (2), let $g: X \to Y$ be a function with a compact graph and let $A \subset Y$ be closed; since π_y is continuous, $\pi_y^{-1}(A) \cap G(g)$ is compact and so the image of this set under π_x is compact; so $g^{-1}(A) = \pi_x[\pi_y^{-1}(A) \cap G(g)]$ is compact in X and thus closed in X. To see that (2) implies (3) let $g: Y \to X$ have a compact graph, $G(g)$. Let $A \subset Y$ be closed. Then $\pi_y^{-1}(A) \cap G(g)$ is compact, so $g(A) = \pi_x(\pi_y^{-1}(A) \cap G(g))$ is compact in X. If T is the topology on X, then X is compact with the simple extension, $T(g(A))$, of T through the compact set $g(A)$ and $g(A)$ is $T(g(A))$-closed (see [**3**]). The identity function i from (X, T) to $(X, T(g(A)))$ has a compact graph since $T \subset T(g(A))$ and $T(g(A)) \subset T(g(A))$ renders the function h from $(X, T(g(A))$ to $X \times X$ defined by $h(x) = (x, x)$ continuous and since $h(X) = G(i)$. Thus i is continuous from (2) so $g(A) = i^{-1}(g(A))$ is T-closed in X. Finally, to verify that (3) implies (1), let $A \subset X$ be compact.

$G(i)$ is compact for the identity function i from $(X, T(A))$ to X (same reasoning as above) so i is closed. Since A is $T(A)$-closed, A is closed in X. This completes the proof.

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Which Nonnegative Matrices are Self-Inverse?

FRANK HARARY
University of Michigan

HENRYK MINC
University of California, Santa Barbara

A binary relation, a digraph, and a graph are represented faithfully by their (square) adjacency matrices which are respectively: binary, binary with zero diagonal, and binary symmetric with zero diagonal. To determine the self-inverse matrices of these three types, we generalize the question to nonnegative matrices and then specialize the result.

THEOREM. *Let A be a nonnegative matrix with the property that $A^2 = I_n$. Then there exists a permutation matrix P such that PAP^T is a direct sum of 1-square matrices* [1] *and 2-square matrices of the form* $A = \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix}$, $a > 0$.

Proof. We first show that if A is an irreducible nonnegative matrix and $A^2 = I_n$, then either $n = 1$ and $A = [1]$, or $n = 2$ and $A = \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix}$ for some positive number a. Recall that by the Perron-Frobenius theorem (see, e.g., Theorem 5.5.1 in [**2**]) all eigenvalues of maximum modulus of an irreducible matrix are distinct. On the other hand, if $A^2 = I_n$ then all the eigenvalues of A are ± 1. It follows that either $n = 1$ and $A = [1]$ or A has exactly two eigenvalues, 1 and -1. In the latter case A is a 2-square nonnegative matrix with trace 0 and determinant -1. Clearly A must be of the form $\begin{bmatrix} 0 & a^{-1} \\ a & 0, \end{bmatrix}$ $a > 0$.

If A is reducible, then there exists a permutation matrix Q such that

$$QAQ^T = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix},$$

where B and C are square. We show that $D = 0$ and thus that the result is provable by induction on n. Now,

$$I_n = A^2 = (QAQ^T)^2 = \begin{bmatrix} B^2 & BD + DC \\ 0 & C^2 \end{bmatrix},$$

and therefore $BD + DC = 0$ and B^2 is an identity matrix. Hence

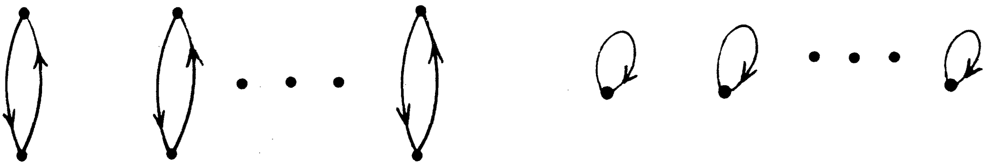
$$0 = BD + DC = B^2D + BDC = D + BDC,$$

which implies that $D = 0$, since all the matrices are nonnegative, completing the proof.

A **relation** R **of order** n is a subset of the cartesian product of $\{1, 2, \ldots, n\}$ with itself. Its **adjacency matrix** $A(R) = (a_{ij})$ has $a_{ij} = 1$ whenever $(i, j) \in R$ and $a_{ij} = 0$ otherwise. A **binary matrix** has entries 0 and 1 only. Thus every square binary matrix of order n is the adjacency matrix of a relation of order n and conversely. The theorem immediately implies the next statement.

COROLLARY. *A square binary matrix A is self-inverse if and only if there exists a permutation matrix P such that PAP^T is a direct sum of 1-square matrices* [1] *and 2-square matrices* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

FIGURE 1 displays the graphical representation (see [**1**]) of such a relation. The main point of the corollary is that the corresponding relation must be symmetric, although this fact is not known *a priori.*



A relation of order n with n_1 symmetric pairs and n_2 loops, where $2n_1 + n_2 = n$.

FIGURE 1

A proof of the corollary (for relations) can be supplied without using all the machinery in the proof of the theorem, because one can exploit the fact that the given matrix is binary.

The case of digraphs simply rules out the presence of loops since a **digraph** is defined as an irreflexive relation. Thus in FIGURE 1 we would have $n_1 = n/2$ symmetric pairs and $n_2 = 0$ loops. Finally, since a **graph** is a symmetric digraph, the resulting adjacency matrices have the same form as for digraphs, but the proof can be accomplished very easily from first principles.

Research for this paper was supported in part by grants 73–2502 and 72–2164 from the Air Force Office of Scientific Research.

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Domains of Dominance

MIRIAM HAUSMAN
The City College of CUNY

Let $f(z)$ be an entire function,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k,$$

$S_n(z)$ the sum of this series for $k \leqq n$, and $R_n(z) = f(z) - S_{n-1}(z)$. Our interest is to determine the regions for which $|R_n(z)|$ is less than or equal to the next term, that is

$$|R_n(z)| = \left| \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \right| \leqq \left| \frac{f^{(n)}(0)}{n!} z^n \right| .$$

This question makes sense for any entire function, but here we solve the problem only for e^z, sin z and cos z.

THEOREM 1. *Consider the function e^z and the regions $\mathcal{R}_n$ defined for $n \geqq 0$ by*

$$(1) \qquad |R_n(z)| \leqq \frac{|z|^n}{n!} .$$

If $\mathcal{H}_-$ denotes the closed left half plane $\mathrm{Re}(z) \leqq 0$ *then $\mathcal{H}_- \subset \mathcal{R}_n$ for all n and $\mathcal{H}_- = \limsup \mathcal{R}_n$ (that is, any point which belongs to infinitely many $\mathcal{R}_n$ must be in $\mathcal{H}_-$).*

Proof. We first show by induction that every point in the closed left half plane satisfies the set of inequalities (1). For $n = 0$, (1) yields $|R_0(z)| = |e^z| \leqq 1$. Writing $z = x + iy$, $|e^z| = e^x \leqq 1$ if $x \leqq 0$.

Assume that for some n (1) is satisfied by every point $z \in \mathscr{H}_-$. It remains to show then that

(2) $$|R_{n+1}(z)| \leqq \frac{|z|^{n+1}}{(n+1)!}$$

for all z such that $\mathrm{Re}(z) \leqq 0$. Since

(3) $$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + R_{n+1}(z),$$

differentiating both sides of (3) with respect to z yields

(4) $$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^{n-1}}{(n-1)!} + \frac{d}{dz} R_{n+1}(z).$$

Equations (3) and (4) imply that

$$\frac{d}{dz} R_{n+1}(z) = R_n(z)$$

or equivalently

(5) $$R_{n+1}(z) = \int_0^z R_n(z) dz.$$

Let the path from 0 to $z = te^{i\theta}$ be a straight line. Then the induction hypothesis together with (5) yields

$$\begin{aligned} |R_{n+1}(z)| &= \left| \int_0^z R_n(z) dz \right| \leqq \int_0^{|z|} |R_n(z)| d|z| \\ &\leqq \int_0^{|z|} \frac{|z|^n}{n!} d|z| = \int_0^{|z|} \frac{t^n}{n!} dt \\ &= \frac{|z|^{n+1}}{(n+1)!}. \end{aligned}$$

Hence $\mathscr{H}_- \subset \mathscr{R}_n$.

Clearly (1) can hold for all n only in the closed left half plane. For if $\mathrm{Re}(z) > 1$, the inequality for $n = 0$, $|e^z| \leqq 1$ is not satisfied. Since

$$R_n(z) = \frac{z^n}{n!} + \frac{z^{n+1}}{(n+1)!} + \frac{z^{n+2}}{(n+2)!} + \cdots$$

(1) is equivalent to

$$\left| 1 + \frac{z}{n+1} + \frac{z^2}{(n+1)(n+2)} + \cdots \right| \leqq 1.$$

Suppose $z = te^{i\theta} \in \limsup \mathscr{R}_n$ where $t > 0$ and $-\pi/2 < \theta < \pi/2$. Then

$$1 + \frac{t \cos \theta}{(n+1)} + \frac{t^2 \cos 2\theta}{(n+1)(n+2)} + \cdots \leqq 1$$

for infinitely many n. This is equivalent to

$$\cos \theta + \frac{t \cos 2\theta}{(n+2)} + \frac{t^2 \cos 3\theta}{(n+2)(n+3)} + \cdots \leqq 0$$

for infinitely many n. This implies $\cos \theta \leqq 0$, contradicting $\cos \theta > 0$. Hence $\limsup \mathscr{R}_n \subset \mathscr{H}_-$ and the proof is complete.

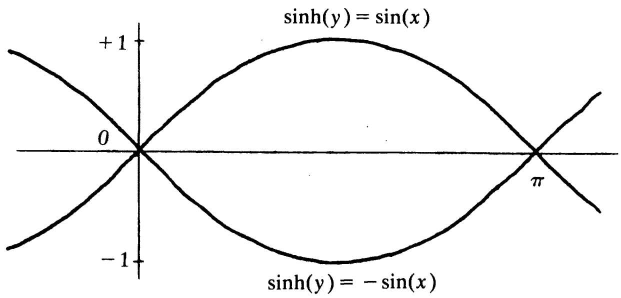

FIGURE 1

THEOREM 2. *Consider the functions* $\sin z$ *and* $\cos z$, *and the regions* $\mathcal{R}_n$ *and* $\tilde{\mathcal{R}}_n$ *defined for* $n \geqq 0$ *by*

$$|R_n(z)| \leqq \frac{|z|^{2n+1}}{(2n+1)!} \quad \text{and} \quad |\tilde{R}_n(z)| \leqq \frac{|z|^{2n}}{(2n)!}, \tag{6}$$

where $R_n(z)$ *and* $\tilde{R}_n(z)$ *correspond to* $\sin z$ *and* $\cos z$ *respectively. Then* $\tilde{\mathcal{R}}_0(z) \subset \mathcal{R}_n(z) \cap \tilde{\mathcal{R}}_n(z)$ *for all* n, *and in fact the only place the set of inequalities* (6) *can hold for all* n *is* $\tilde{\mathcal{R}}_0(z)$.

Proof. It is necessary to show that every point $z \in \tilde{\mathcal{R}}_0(z) = \{z \mid |\cos z| \leqq 1\}$ satisfies the set of inequalities (6) for all n. Let z^* be a point in the region $\mathcal{C}$ defined by $|\cos z| \leqq 1$ and $0 \leqq \mathrm{Re}(z) \leqq \pi$. Consider a straight line path from 0 to $z^* = t^* e^{i\theta}$. Since $\mathcal{C}$ is convex (it is bounded by the curves $\sinh y = \pm \sin x$; see FIGURE 1), this path lies entirely in $\mathcal{C}$. Subdivide this path into $2n+1$ subintervals by inserting the points $z_1, z_2, \ldots, z_{2n+1} = z^*$ where $z_v = t_v e^{i\theta}$, then

$$\begin{aligned} |R_n(z^*)| &= \left| \int_0^{z_{2n+1}} \cdots \int_0^{z_2} \int_0^{z_1} \operatorname{cis} z \, dz dz_1 \, dz_2 \ldots dz_{2n} \right| \\ &\leqq \int_0^{|z_{2n+1}|} \cdots \int_0^{|z_1|} |\cos z| \; d|z| \, d|z_1| \ldots d|z_{2n}| \\ &\leqq \int_0^{t^*} \cdots \int_0^{t_1} 1 \, dt \ldots dt_{2n} \\ &= \frac{t^{*2n+1}}{(2n+1)!} = \frac{|z^*|^{2n+1}}{(2n+1)!}. \end{aligned}$$

Similarly subdividing the path from 0 to z^* into $2n$ subintervals, yields

$$\begin{aligned} |\tilde{R}_n(z^*)| &= \left| \int_0^{z_{2n}} \cdots \int_0^{z_2} \int_0^{z_1} \cos z \, dz dz_1 \ldots dz_{2n-1} \right| \\ &\leqq \frac{|z^*|^{2n}}{(2n)!}. \end{aligned}$$

Clearly if $z \in \tilde{\mathcal{R}}_0(z)$, $\mathrm{Re}(z) \notin [0, \pi]$, then $z = z^* \pm n\pi$ where $0 \leqq \mathrm{Re}(z^*) \leqq \pi$. Since $|\cos z| = |\cos(z^* \pm n\pi)| = |\cos z^*|$ and $|\sin z| = |\sin(z^* \pm n\pi)| = |\sin z^*|$ the set of inequalities (6) is now satisfied by all $z \in \tilde{\mathcal{R}}_0(z)$.

In conclusion, we see that the only place all the inequalities (6) can hold is in the region $\tilde{\mathcal{R}}_0(z)$. For if $|\cos z| > 1$, the inequality when $n = 0$, $|\tilde{R}_0(z)| = |\cos z| \leqq 1$ is not satisfied.

PROBLEMS

DAN EUSTICE, Editor
LEROY MEYERS, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before October 1, 1976.

970. A plus or minus sign is assigned randomly to each of the numbers $1, 2, 3, \ldots, n$. What are the probabilities that the sum of the signed numbers is positive, negative, and zero? [*Martin Berman, Bronx Community College.*]

971. In designing pipes and other conduits it is usually desirable to enclose the maximum cross-sectional area for a given weight of pipe. Mathematically, this may be simplified by enclosing the maximum area for a given perimeter.

Dual ducts are often used to convey fluids in two directions. They have a portion of their perimeter in common. For example, two equal squares, each of side s are placed to share a common side. The total perimeter is $7s$ and the total cross-sectional area is $2s^2$. Thus, the ratio of the area to the square of the perimeter is 2/49. Assume equal cross-sectional area of the two ducts.

(i) Which regular polygon is the most efficient for use as a dual duct?

(ii)* Which contour is the most efficient for use as a dual duct?

[*Sidney Kravitz, Dover, New Jersey.*]

972. Prove or disprove that the set of all positive rational numbers can be arranged in an infinite sequence, $\{a_n\}$, such that $\{(a_n)^{1/n}\}$ is convergent. [*Marius Solomon, Student, University of Pennsylvania.*]

973*. Let N be an odd number; if the period of N^{-1} is P in base b, and if $N^2 \mid b^P - 1$, then the period of N^{-n} in base b is PN^{n-1}. [*Robert Cranga, University of Paris.*]

974. Let $n^{[i]} = n(n-1)\cdots(n-i+1)$. For k a positive integer, evaluate

$$\operatorname*{Limit}_{n\to\infty} \sum_{i=1}^{n} \frac{n^{[i]}}{(2n+k)^{[i]}}.$$

[*John P. Hoyt, Indiana, Pennsylvania.*]

We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has a succinct unexpected solution. Readers desiring acknowledgement of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

975. In FIGURE 1, n, the length of the base, is 5 units, and $f(n)$, the number of different regular hexagons, is 6. Find a formula for $f(n)$. [*Charles L. Hamberg, Prairie View, Illinois, and Thomas M. Green, San Pablo, California.*]

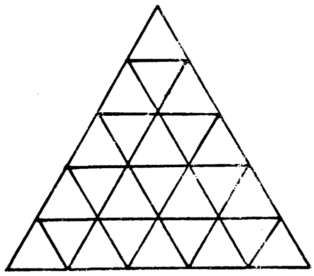

FIGURE 1

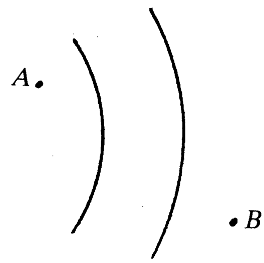

FIGURE 2

976. A road is to be built connecting two towns separated by a river whose banks are concentric circular arcs (see FIGURE 2). If the road must bridge the river banks orthogonally, describe the minimum length road (assuming coplanarity). [*Miller Puckette and Steven Tschanz, 1975 U.S.A. International Mathematical Olympiad Team.*]

977. Let x and y be two integers with $1 < x < y$ and $x + y \leqq 100$. Suppose Ms. S. is given the value of $x + y$ and Mr. P. is given the value of xy.

(1) Mr. P. says: "I don't know the values of x and y."
(2) Ms. S. replies: "I knew that you didn't know the values."
(3) Mr. P. responds: "Oh, then I do know the values of x and y."
(4) Mr. S. exclaims: "Oh, then so do I."

What are the values of x and y? [*David J. Sprows, Villanova University.*]

Editor's Comment. Proposal 977 is a succinct variation of some past problems in the Amer. Math. Monthly, especially E776, E1126, and E1156.

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q633. The sum of the first eight positive integers is 36, a perfect square. Are there any other values of k for which the sum of the first k positive integers is a perfect square? Are there infinitely many k? [*J. D. Baum, Oberlin College.*]

Q634. If a, b, c, d are positive integers where $ab = cd$, *show that* $a^2 + b^2 + c^2 + d^2$ is always composite. [*M. S. Klamkin, University of Waterloo.*]

Solutions

Two Octahedrons **March 1975**

929. Show that there are only two octahedrons with equilateral triangular faces. [*Charles W. Trigg, San Diego, California.*]

Solution: Any octahedron must obey Euler's Formula and as these are to have triangular faces, the only solution would be 8 faces, 12 edges and 6 vertices. Considering the connectivity of the faces at each of the vertices, observe that each vertex must be adjacent to 3, 4 or 5 faces. No other possibility exists due to the sum of the angles involved. The sum of the connectivities at the vertices also must be 24 (8 faces each with 3 vertices). Thus there are only four possible combinations of connectivity at the vertices. These are 4–4–4–4–4–4, 3–3–4–4–5–5, 3–4–4–4–4–5 or 3–3–3–5–5–5. The graphs corresponding to these four situations may be sketched and the first two cases easily found to be planar. The second two cases have the following adjacency graphs.

Eliminating the dashed edges, both graphs are seen to contain $K_{3,3}$ and thus be nonplanar by Kuratowski's theorem. Therefore the only possible octahedrons with equilateral triangular faces are the 4–4–4–4–4–4 regular and the 3–3–4–4–5–5 "tent".

Larry M. Hopkins
Gogebic Community College

Also solved by Michael Goldberg and the proposer.

A System of Equations **March 1975**

930. Solve the system of equations $(x_i - a_{i+1})(x_{i+1} - a_{i+3}) = a_{i+2}^2$, $i = 1, 2, \ldots, n$, for the x_i's where $a_{n+i} = a_i$, $x_{n+i} = x_i$, and $a_1 a_2 \cdots a_n \neq 0$. [*M. S. Klamkin, University of Waterloo.*]

Solution: Let $y_i = x_i - a_{i+1}$ $(\neq 0)$, $i = 1, 2, \ldots, n$. The given equations reduce to

$$(*) \qquad y_{i+1} - a_{i+3} = -\frac{a_{i+2}}{y_i}(y_i - a_{i+2}),$$

$$y_i = y_{n+i}, \qquad i = 1, 2, \ldots, n.$$

Suppose $y_i = a_{i+2}$ for some $i = 1, 2, \ldots n$; then $y_i = a_{i+2}$ for all $i = 1, 2, \ldots, n$. A solution is therefore furnished by $y_i = a_{i+2}$, i.e.,

$$x_i = a_{i+1} + a_{i+2}, \qquad i = 1, 2, \ldots, n.$$

If $y_i \neq a_{i+2}$ for all $i = 1, 2, \ldots, n$, then we have

$$\Pi_i y_i = (-1)^n \Pi_i a_i.$$

This indicates that we may have solutions of the form

$$y_i = a_{i+2} + k_i,$$

where $k_i \neq 0$, $k_i = k_{n+i}$, $i = 1, 2, \ldots, n$. $(*)$ gives

$$(**) \qquad \frac{1}{k_i} + \frac{1}{k_{i+1}} = -\frac{1}{a_{i+2}}.$$

We distinguish between the following two cases:

(1) n is odd. From $(**)$

$$\frac{2}{k_i} = \left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right) - \left(\frac{1}{k_{i+1}} + \frac{1}{k_{i+2}}\right) + \cdots$$
$$- \left(\frac{1}{k_{i-2}} + \frac{1}{k_{i-1}}\right) + \left(\frac{1}{k_{i-1}} + \frac{1}{k_i}\right)$$
$$= -\frac{1}{a_{i+2}} + \frac{1}{a_{i+3}} - \cdots + \frac{1}{a_i} - \frac{1}{a_{i+1}},$$
$$k_i = 2\left\{\sum_{p=2}^{n+1} \frac{(-1)^{p+1}}{a_{i+p}}\right\}^{-1}.$$

(2) n is even. $(**)$ is consistent if and only if $\Sigma_{i=1}^{n}((-1)^i/a_i) = 0$. Under this condition, we have

$$\frac{1}{k_i} = (-1)^{i+1}\lambda + \frac{1}{n}\sum_{p=1}^{n-1} \frac{(-1)^p(n-p)}{a_{i+p-1}},$$

where λ is a parameter. (A standard algorithm leads to a solution in which $x_n = \lambda_n$ is a parameter; the above expression is obtained by symmetrizing this parametric solution.) Thus, if n is even and if

$$\sum_{i=1}^{n} \frac{(-1)^i}{a_i} = 0,$$

the given system admits, besides the obvious solution $x_i = a_{i+1} + a_{i+2}$, an infinite number of solutions given by

$$x_i = a_{i+1} + a_{i+2} + \left\{(-1)^{i+1}\lambda + \frac{1}{n}\sum_{p=1}^{n-1} \frac{(-1)^p(n-p)}{a_{i+p+1}}\right\}^{-1},$$

where λ is an arbitrary parameter except that it cannot be chosen to make any $1/k_i$ vanish.

Paul Y. H. Yiu
University of Hong Kong

Also solved by J. C. Binz (Switzerland), Thomas Elsner, Richard A. Groeneveld, Temple University Problem Solving Group, and the proposer. Partial solutions by Michael Goldberg and Robert M. Hashway.

Logically Speaking **March 1975**

931. For each $r \leqq n$ in a list of n statements, the rth statement is: "The number of false statements in this list is greater than r." Determine the truth value of each statement. [*Alan Wayne, Holiday, Florida.*]

Solution: Let p be the number of false statements. Then the first $p-1$ statements are true and the last p statements are false. This implies that n must be odd. To see this another way suppose n is even. Let p be as above, t equal the number of true statements and S_i be statement i.

Case I. If $t = p$, then $S_{n/2}$ is true. But $S_{n/2}$ would say $p > n/2$, a contradiction.

Case II. If $t > p$, then $S_{n/2+1}$ is true. But $S_{n/2+1}$ states $p > n/2+1$, and so $t > p > n/2+1$. Hence $t + p > 2(n/2+1) > n$. This contradicts the fact that $p + t = n$.

Case III. If $t < p$, then $S_{n/2-1}$ is false. Then $p \leqq n/2-1$ and so $t < p \leqq n/2-1$. Hence $t + p < 2(n/2-1) < 2n$. Contradiction. So n is not even.

RAYMOND A. MARUCA
Delaware County Community College

Also solved by Clayton W. Dodge, Michael W. Ecker, Thomas E. Elsner, Gino T. Fala, Donald G. Fuller, Richard A. Gibbs & William C. Ramaley, Michael Goldberg, Ronald Gutman, Chandra M. R. Kinalta, N. J. Kuenzi & John Oman, Mary Helen Manning, William Nuesslein, C. B. A. Peck, Daniel Mark Rosenblum, Joseph Silverman, Brian Smithgall, Temple University Problem Solving Group, G. Wedderburn, Kathy Woerner, Ken Yocom, Y. H. Yiu (Hong Kong), and the proposer.

Connected Sets **March 1975**

932. Is there a topology for the set of real n-tuples other than the Euclidean topology, relative to which the family of connected sets is exactly the usual one? [*R. A. Struble, N. C. State University at Raleigh.*]

Solution: Yes. Let $\mathscr{E}$ be the Euclidean topology on $\mathbf{R}^n$, and define $\mathscr{B}$ by $\mathscr{B} = \mathscr{E} \cup \{U \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) \mid U \in \mathscr{E}\}$. $\mathbf{Q}$ is the set of rational numbers. Now $\mathbf{R}^n \in \mathscr{B}$ and $B_1, B_2 \in \mathscr{B}$ implies that $B_1 \cap B_2 \in \mathscr{B}$. Hence there exists a unique topology $\mathscr{T}$ on $\mathbf{R}^n$ such that $\mathscr{B}$ is a base for $\mathscr{T}$. Note that $\mathscr{E} \subset \mathscr{T}$ but $\mathscr{E} \neq \mathscr{T}$.

If $A \subset \mathbf{R}^n$ is $\mathscr{E}$-disconnected, then A is also $\mathscr{T}$-disconnected, since $\mathscr{E} \subset \mathscr{T}$.

Conversely, suppose $A \subset \mathbf{R}^n$ is $\mathscr{T}$-disconnected. Let $\mathscr{T}_A$, $\mathscr{E}_A$ be the subspace topologies on A induced by $\mathscr{T}$, $\mathscr{E}$ and let $\{V, W\} \subset \mathscr{T}_A$ be a $\mathscr{T}_A$-separation of A. If $\{V, W\} \subset \mathscr{E}_A$, then A is $\mathscr{E}$-disconnected and we're done. So suppose without loss of generality that $V \not\in \mathscr{E}_A$. Then there exists an $x \in V$ such that $x \in U$ implies that $U \cap W \neq \varnothing$ for each $U \in \mathscr{E}$. However, there exists a $U^* \in \mathscr{E}$ such that $x \in U^* \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) \cap A \subset V$. Note that $U^* \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) \cap W \neq \varnothing$.

Pick $y \in U^* \cap W$. Using the facts that $U^* \cap A \in \mathscr{E}_A$ and $\mathscr{E}_A \subset \mathscr{T}_A$ we may conclude that $U^* \cap W = U^* \cap A \cap W \in \mathscr{T}_A$. Hence there exists a $B \in \mathscr{B}$ such that $y \in B \cap A \subset U^* \cap W$. Now B cannot be of the type $U \cap (\mathbf{R}^n \backslash \mathbf{Q}^n)$ for some $U \in \mathscr{E}$, because we've already noted that $U^* \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) \cap W = \varnothing$. Hence $B \in \mathscr{E}$.

Now select irrationals $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ such that the n-cell $S = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \mid u_i \leqq x_i \leqq v_i \text{ for } i = 1, 2, \ldots, n\}$ satisfies:

(i) $S \subset B$;

(ii) $y \in \operatorname{int}_{\mathscr{E}} S = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \mid u_i < x_i < v_i \text{ for } i = 1, 2, \ldots, n\}$.

Note that $(x_1, x_2, \ldots, x_n) \in \operatorname{bdy}_{\mathscr{E}} S$ implies x_i is irrational for some i, $1 \leqq i \leqq n$, which implies $(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \backslash \mathbf{Q}^n$.

Hence $\operatorname{bdy}_{\mathscr{E}} S \subset \mathbf{R}^n \backslash \mathbf{Q}^n$, so that $(\operatorname{bdy}_{\mathscr{E}} S) \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) = \operatorname{bdy}_{\mathscr{E}} S$.

So

$$\begin{aligned}(\operatorname{bdy}_{\mathscr{E}} S) \cap A &= (\operatorname{bdy}_{\mathscr{E}} S) \cap A \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) \\ &\subset B \cap A \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) \\ &\subset U^* \cap W \cap (\mathbf{R}^n \backslash \mathbf{Q}^n) = \varnothing.\end{aligned}$$

Therefore

$$\begin{aligned}A \cap S &= A \cap (\operatorname{int}_{\mathscr{E}} S \cup \operatorname{bdy}_{\mathscr{E}} S) \\ &= [A \cap (\operatorname{int}_{\mathscr{E}} S)] \cup [A \cap \operatorname{bdy}_{\mathscr{E}} S] \\ &= A \cap (\operatorname{int}_{\mathscr{E}} S) \in \mathscr{E}_A.\end{aligned}$$

Note that $y \in A \cap S$. Also $x \in A \backslash S = A \cap (\mathbf{R}^n \backslash S) \in \mathscr{E}_A$.

So $\{A \cap S, A \setminus S\} \subset \mathscr{C}_A$ is an $\mathscr{C}_A$ separation of A and A is $\mathscr{C}$-disconnected.

TOM GEARHART, Graduate Student
The Ohio State University

Also solved by the proposer.

The Limit is *e* **March 1975**

933. Show that

$$\lim_{n\to\infty} \frac{n^2}{(1\cdot 2^2\cdot 3^3\cdots n^n)^{4/n^2}} = e.$$

[*Norman Schaumberger, Bronx Community College.*]

Solution: In view of the identity

$$(n\cdot n^2\cdot n^3\cdots n^n)^{4/n^2}\cdot n^{-2/n} = n^2$$

and the well-known limit relation $\lim_{n\to\infty} n^{1/n} = 1$, we have only to prove

$$\left(\frac{n\cdot n^2\cdot n^3\cdots n^n}{1\cdot 2^2\cdot 3^3\cdots n^n}\right)^{4/n^2} \equiv P_n \text{ tends to } e \text{ as } n\to\infty.$$

Taking the natural logarithm, we see

$$\log P_n = -\frac{4}{n^2}\sum_{k=1}^{n} k\log\frac{k}{n} = -\frac{4}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n}$$

which in turn tends to $-4\int_0^1 x\log x\,dx = 1$ as $n\to\infty$.

KOICHI SATO, Student
Tôhoku University (Japan)

Also solved by M. Ahuja & Leonard Palmer, Melven Billik, J. C. Binz (Switzerland), M. T. Bird, C. S. Brocato & J. M. Stark, Roy E. DeMeo, Irwin Feinstein, Stanley Fox, Ralph Garfield, Leon Gerber, M. G. Greening (Australia), Richard A. Groeneveld, Shyam Johari, Steven Kahan, Carl P. McCarty, William Nuesslein, T. K. PuttaSwamy, Richard Renner, Harry D. Ruderman, Joseph Silverman, F. G. Schmitt, Jr., L. Van Hamme (Belgium), B. Watson (Canada), Ken Yocom, and the proposer.

A Subset of Integers **March 1975**

934. From the first kn positive integers, choose a subset, K, consisting of $(k-1)n+1$ distinct integers. Prove that at least one member of K is the sum of k members (not necessarily distinct) of K. [*Erwin Just, Bronx Community College.*]

Solution: Note that $kn-[(k-1)n+1] = n-1$ integers are missing from K. Consider $2n$ integers paired into n pairs, thus:

$$(1,k),(2,2k),(3,3k),\ldots,(n-1,(n-1)k),(n,nk).$$

Each of these $2n$ integers is among the set of the first kn positive integers, and at most $n-1$ of these $2n$ integers are missing from K. Since there are n such pairs, at least one complete pair remains in K. But the second element in any such pair is the sum of k of the first pair, and hence at least one member of K is the sum of k members (definitely not distinct in this construction) of K.

DONALD C. FULLER
Gainesville Junior College

Also solved by J. C. Binz (Switzerland), Richard Gibbs, Michael Goldberg, Graham Lord, J. M. Stark, G. Wedderburn, Kenneth M. Wilke, and the proposer.

936.* It is known that $h_a + h_b + h_c \leqq \sqrt{3}s$, where the h's represent altitudes to sides a, b, and c and s represents the semiperimeter of triangle ABC. Prove or disprove the stronger inequality $t_a + t_b + m_c \leqq \sqrt{3}s$, where the t's are the angle bisectors and m_c is the median to side c. [*Jack Garfunkel, Flushing, New York.*)

Solution: Upper and lower bounds of $m_a + m_b + m_c$ are $2s$ and $3s/2$, respectively. When the sum of the medians lies between $3s/2$ and $s\sqrt{3}$, the result $t_a + t_b + m_c \leqq m_a + m_b + m_c \leqq s\sqrt{3}$ is trivial, equality holding if and only if the triangle is equilateral. We may therefore focus our attention on those cases where the sum of the medians exceeds $s\sqrt{3}$.

Since $4bc \leqq (b+c)^2$ with equality if and only if $b = c$ (with a similar inequality connecting a and c), it is apparent that $m_a[4bc/(b+c)^2] + m_b[4ac/(a+c)^2] + m_c \leqq m_a + m_b + m_c$, equality holding only when $a = b = c$, that is, when the triangle is equilateral and $m_a + m_b + m_c = s\sqrt{3}$. Consequently the left side of the inequality attains its maximum value of $s\sqrt{3}$ only when $m_a = m_b = m_c$ and remains less than $s\sqrt{3}$ when the triangle is not equilateral.

By the result of problem E 2471 [Amer. Math. Monthly, 1974, 406], it is known that $t_a \leqq m_a[4bc/(b+c)^2]$ and that $t_b \leqq m_b[4ac/(a+c)^2]$, with equality only when $b = c$ and $a = c$ respectively. We may therefore write the chain of inequalities

$$t_a + t_b + m_c \leqq m_a[4bc/(b+c)^2] + m_b[4ac/(a+c)^2] + m_c \leqq s\sqrt{3}. \tag{1}$$

When $m_a + m_b + m_c$ exceeds $s\sqrt{3}$, the triangle cannot be equilateral and the chain of inequalities (1) remains valid since the central sum cannot exceed $s\sqrt{3}$.

For example, if $a = 17$, $b = 8$ and $c = 15$, the sum of the lengths of the medians is 34.99+, $s\sqrt{3} = 34.64+$, while $m_a[4bc/(b+c)^2] + m_b[4ac/(a+c)^2] + m_c = 34.14+$ and $t_a + t_b + m_c = 33.80+$. If m_a is the unweighted median, the sum is 33.508+, corresponding to $m_a + t_b + t_c = 33.29+$. If m_b is unweighted, the sum is 32.78+ while $t_a + m_b + t_c$ is equal to 32.23+. Similar examples can easily be constructed.

LEON BANKOFF
Los Angeles, California

Also solved by W. A. Sentance (England).

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q633. If the sum is a square, say n^2, then $k^2 + k - 2n^2 = 0$. Using the quadratic formula, since $k > 0$, we have $k = (-1 + \sqrt{(1+8n^2)})/2$. Since k is an integer, there is an integer p such that $1 + 8n^2 = p^2$, or $p^2 - 2(2n)^2 = 1$. This is a Pell equation which may be solved by standard means (see, for example, Niven and Zuckerman, *An Introduction to the Theory of Numbers*, 3rd ed., 1972, 172–175). Thus, triples (n, p, k): (6, 17, 8), (35, 99, 49), (204, 577, 288), etc., generate solutions. Since the Pell equation has infinitely many solutions, we find infinitely many k and n which solve the problem.

Q634. Since $d = ab/c$, $a = mn$, $b = rs$, $c = mr$, $d = ns$. Then,

$$a^2 + b^2 + c^2 + d^2 = (m^2 + s^2)(n^2 + r^2).$$

This problem appeared on a West German Olympiad.

NEWS & LETTERS

BULK SUBSCRIPTIONS

To make it more attractive for mathematics departments to place *Mathematics Magazine* in the hands of their better undergraduates, the M.A.A. has authorized a bulk subscription rate of $3.00 per year for orders of five or more copies to a single address. Orders must be placed by department chairmen on official letterhead, must state that the copies are to be used by undergraduate students, and must include payment or a purchase order. Bulk subscriptions must be for one year beginning either in January or in September and should be sent to: The Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D.C. 20036.

NINE ODD RECIPROCALS

In his "Expressing One as a Sum of Odd Reciprocals" (this *Magazine*, January 1976, p. 31), E.J. Barbeau expressed 1 as a sum of 13 distinct odd reciprocals, and asks if 11 or 9 is possible. Nine is indeed possible; a computer program revealed five such solutions. All five include {3,5,7,9, 11,15}. The other three elements can be {21,135,10395} , {21,165,693} , {21,231,315}, {33,45,385} , or {35,45, 231}. The program took 0.356 seconds to compile and execute on a CDC 7600.

Peter L. Montgomery
System Development Corp.
Huntsville
Alabama 35805

TRISECTOR REFERENCES

The angle trisector mentioned in George B. Miller's letter (this *Magazine*, January 1976, pp. 50-51) was first developed about a century ago by a London barrister, Alfred Bray Kempe. Kempe's angle trisector is mentioned, for example, in Joseph Hilsenrath's "Linkages" in *The Mathematics Teacher*, October 1937, pp. 277-284, and also in my "Linkages as Visual Aids" in *The Mathematics Teacher*, December 1946, pp. 277-284.

Bruce E. Meserve
Fairfax
Vermont 05454

TRANSPOSABLE INTEGERS

In Steven Kahan's interesting article (this *Magazine*, January 1976, pp. 27-28) his "k-transposable" integers are "k-right-transposable" since each is formed by transferring the leftmost digit to the right to become the units digit. A "k-left-transposable" integer is formed by transferring the units digit x_0 to the left to become the leftmost digit. Kahan has shown that all k-right-transposable integers are obtained from the decimal expansions of the two fractions 1/7 and 2/7; namely, $\overline{142857}_m$ and $\overline{285714}_m$. It can be shown that all k-left-transposable integers are obtained from the decimal expansions of the thirty-six fractions defined by $x_0/(10k - 1)$, in which $1 < k \le x_0 < (10k - 1)/k$. For example, if $x_0 = 6$ and $k = 4$, the fraction 6/39 yields $\overline{153846}_m$, which multiplied by 4 becomes $\overline{615384}_m$.

Alan Wayne
Holiday
Florida 33589

A SHORTER PROOF

Here is a shorter derivation of the forms of the "curious sequence" discussed by Steven Kahan (this *Magazine*, November 1975, pp. 290-291). The number k is at the same time the number of members of the sequence $\{n_i\}$ and the sum of these members:

$$k = \sum_{i=0}^{k-i} n_i = \sum_{i=1}^{k-1} in_i \quad .$$

By subtraction

$$(*) \qquad n_0 = \sum_{i=2}^{k-1} (i-1)n_i \ .$$

If $n_0 = j \geq 3$, (*) implies $n_j = 1$, $n_2 = 1$, and $n_i = 0$ for $i \geq 3$, $i \neq j$; therefore $n_1 = 2$ and $k = j + 4 \geq 7$. For each $k \geq 7$ there is an unique solution $(k-4,2,1,0,\ldots,0,1,0,0,0)$.

Let $n_0 = 2$. Now $n_2 \geq 1$, and with (*) follows that $n_2 = 2$, $n_i = 0$ for $i \geq 3$. There are two solutions (2,0,2,0) and (2,1,2,0,0).

If, finally, $n_0 = 1$, (*) gives $n_2 = 1$, $n_i = 0$ for $i \geq 3$, and hence $n_1 = 2$, $k = 4$; the solution is (1,2,1,0).

J.C. Binz
Universität Bern
Bern, Switzerland

A BIG MEMORY

Charles J. Mifsud (this *Magazine*, May 1975, pp. 174-176) concludes in his paper "On the Representation of a Possible Solution Set of Fermat's Last Theorem" that to discover a solution set for Fermat's Last Theorem, a computer would have to deal with numbers having almost 188 million digits of precision.

My father established in 1934 (*Amer. Math. Monthly*, 41 (1934) 419-424) that Fermat's equation $x^n + y^n = z^n$ cannot be satisfied by integers x, y, z, n, where $x, y, z \not\equiv 0 \mod n$, unless the smallest of x, y, z exceeds $n(2n+1)^n$. Accepting $n = 25{,}000$, as cited by Mr. Mifsud, one needs a memory with over 2,900,000,000 base 10 digits. Is there a reference for larger sized requirements where n divides one of x, y, or z?

Glenn D. James
Los Angeles City Coll.
Los Angeles
California 90029

WHAT DID HAMILTON KNOW?

Professor Schenkman's article "A group theoretic presentation of the alternating group on five symbols, A_5," (this *Magazine*, May 1975, pp. 170-171) seems to be a rediscovery. On page 138 of Coxeter and Moser's *Generators and Relations for Discrete Groups* (Springer, 1957), the definition $R^3 = S^3 = (RS)^5 = 1$ is attributed to W.R. Hamilton, *Phil. Mag.* (4) 12 (1856) p. 856. Just set $a = S^{-1}R^{-1}$, $b = R^{-1}$, and (since $S^{-1} = ab$) you have Schenkman's proposition. Indeed, Hamilton would have written $R^{-2} = S^{-3} = (RS)^{-5} = 1$ if he had anticipated what Schenkman was going to discover!

Joel Brenner
Palo Alto
California 94303

Professor Brenner apparently fails to appreciate the fact that I prove in my article that $PL(2,5) \cong A_5 \cong G(a,b \mid a^5 = 1 = b^2 = (ab)^3)$. This is not done either in Coxeter and Moser, or by Hamilton; in fact the notion of a group in terms of generators and relations was probably not formalized till this century, though the old masters certainly understood more than they wrote down.

Eugene Schenkman
Purdue University
Lafayette
Indiana 47907

RECOUNTING LINES

The "counting line" technique for plane dissection problems featured on your January cover and utilized by J.W. Freeman in his note "The Number of Regions Determined by a Convex Polygon" (this *Magazine*, January 1976, pp. 23-25) is due to Bro. Alfred Brousseau ("A Mathematician's Progress", *Math. Teacher* 59 (1966) 722-727) and is now fairly well know. Brousseau's method solves, not to say elegantly, many interesting partition problems. Using Brousseau's technique, Jeanne W. Kerr and I have recently determined the number of cells formed in space by the (extended) face-planes of the Platonic solids; and G.L. Alexanderson and I as well as Curtis Greene and Thomas

Zaslavsky have obtained various analogs and generalizations in higher dimensions.

The problem of counting the regions formed by n nonparallel lines that meet to form W_i points of multiplicity i (for $2 \leq i \leq n$), solved by Freeman on page 24, is also nearly a decade old, having been posed by J.A. Burslem (E1869, *Amer. Math. Monthly* 73 (1966) 309) and solved by Harley Flanders (*Amer. Math. Monthly* 74 (1967) 864-865).

John E. Wetzel
University of Illinois
Urbana
Illinois 61801

1, 2, 4, 8, 16, ...

The application of Euler's Formula to Pick's Theorem given in the article "Triangulation and Pick's Theorem" (this *Magazine*, January 1976, pp. 35-37) should serve as a model for the mathematician's peculiar use of the word "elegant." On the margins of this article I have scribbled myself the following note: Use this technique to determine the region count N discussed on page 23. For that count the 2-copy spherical configuration has $n + 2\binom{n}{4}$ vertices and $\binom{n}{2} + 4\binom{n}{4}$ edges. (Each of the $\binom{n}{4}$ interior points interrupts 2 of the $\binom{n}{2}$ chords producing--as doubled on the sphere--4 additional

ANSWERS AND HINTS FOR . . .

In January we printed in this column the questions from the 36th William Lowell Putnam Examination, administered on December 6, 1975. To assist those readers who have been puzzling over these problems for the past two months, we provide here answers and hints as prepared by the Putnam examination committee. The official report on the results of the competition, including names of the winners and complete sample solutions, will be published late this year in the American Mathematical Monthly.

A-1. Supposing that an integer n is the sum of two triangular numbers, $n = \frac{1}{2}(a^2 + a) + \frac{1}{2}(b^2 + b)$, write $4n + 1$ as the sum of two squares, $4n + 1 = x^2 + y^2$, and show how x and y can be expressed in terms of a and b. Show that, conversely, if $4n + 1 = x^2 + y^2$, then n is the sum of two triangular numbers.

Ans. $x = a + b + 1$, $y = a - b$. For the converse, exactly one of x and y is odd; then the desired integers are $a = (x+y-1)/2$ and $b = (x-y-1)/2$.

A-2. For which ordered pairs of real numbers b, c do both roots of the quadratic equation $z^2 + bz + c = 0$ lie inside the unit disk $\{|z| < 1\}$ in the complex plane?

Ans. The desired region in the bc-plane is the inside of the triangle bounded by the lines $L_1: c = 1$, $L_2: c - b + 1 = 0$, and $L_3: c + b + 1 = 0$. Let the roots of the quadratic be r and s. Above, on, and below L_1 one has $|rs| > 1$, $|rs| = 1$, and $|rs| < 1$, respectively. Below, on, and above L_2 one has $(r+1)(s+1)$ less than, equal to, and greater than 0, respectively. Below, on, and above L_3 one has $(r-1)(s-1)$ less than, equal to, and greater than 0, respectively.

A-3. Let a, b, c be constants with $0 < a < b < c$. At what points of the set $\{x^b + y^b + z^b = 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$ in three-dimensional space R^3 does the function $f(x, y, z) = x^a + y^b + z^c$ assume its maximum and minimum values?

Ans. Let $x_0 = (a/b)^{1/(b-a)}$ and $z_0 = (b/c)^{1/(c-b)}$. The point $(x_0, [1 - x_0^b]^{1/b}, 0)$ maximizes both $x^a - x^b$ and $z^c - z^b$ and so maximizes f subject to $x^b + y^b + z^b = 1$. Similarly, $(0, [1 - z_0^b]^{1/b}, z_0)$ gives the minimum. There are no other solutions.

A-4. Let $n = 2m$, where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Express $(1 - \theta)^{-1}$ explicitly as a polynomial in θ,

$$a_k\theta^k + a_{k-1}\theta^{k-1} + \dots + a_1\theta + a_0,$$

with integer coefficients a_i.

edges.) Substitution into Euler's Formula gives $N = \binom{n}{4} + \binom{n-1}{2}$.

If this polygonal configuration is placed in a circle, adding n regions, the new count is given by $a_n = N + n = \binom{n}{4} + \binom{n}{2}$. The nice thing about the sequence $\{a_n\}$ --especially since we now have such a neat way to get at its formula--is that it gives an example showing how untrustworthy "patterns" can be, for its first six terms are 1, 2, 4, 8, 16, 31.

John Staib
Drexel University
Philadelphia
Pennsylvania 19104

PERIODIC PROOFS

Readers interested in the theorem that a measurable function with arbitrarily small periods is equivalent to a constant (H. Burkill, this *Magazine*, September 1974, pp. 206-210) might like to know of the existence of some other proofs: R. Hartman and R. Kershner, *Amer. J. Math.*, 59, 809-822 (1937) (p. 815); R.P. Boas, *Nieuw Archief voor Wiskunde* (3) 5 (1957), 25; (3) 1 (1953), 27-32.

R.P. Boas
Northwestern Univ.
Evanston
Illinois 60201

...THE 1975 PUTNAM EXAMINATION

Ans. Let $k = (n-2)/4$. Then $(1-\theta)^{-1} = \theta + \theta^3 + \theta^5 + \ldots + \theta^{2k-1} = 1 + \theta^2 + \theta^4 + \ldots + \theta^{2k}$ as one can show using $\theta^{2k+1} - 1 \neq 0$, $\theta + 1 \neq 0$, and $0 = \theta^n - 1 = (\theta^{2k+1} - 1)(\theta + 1)(\theta^{2k} - \theta^{2k-1} + \theta^{2k-2} - \ldots - \theta + 1)$.

A-5. On some interval I of the real line, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation $y'' = f(x)y$, where $f(x)$ is a continuous real-valued function. Suppose that $y_1(x) > 0$ and $y_2(x) > 0$ on I. Show that there exists a positive constant c such that, on I, the function $z(x) = c\sqrt{y_1(x)y_2(x)}$ satisfies the equation $z'' + z^{-3} = f(x)z$. State clearly the manner in which c depends on $y_1(x)$ and $y_2(x)$.

Ans. The answer for c is $\sqrt{2/w}$, where w is the Wronskian $y_1y_2' - y_2y_1'$, which is constant since $w' = y_1y_2'' - y_2y_1'' = y_1fy_2 - y_2fy_1 = 0$.

A-6. Let P_1, P_2, P_3 be the vertices of an acute-angled triangle situated in three-dimensional space. Show that it is always possible to locate two additional points P_4 and P_5 in such a way that no three of the points are collinear and so that the line through any two of the five points is perpendicular to the plane determined by the other three.

Ans. Let i, j, k be any permutation of 1, 2, 3 and let Q_i be the foot of the perpendicular from P_i to line P_jP_k. Let H be the orthocenter of $\Delta P_1P_2P_3$, i.e., intersection of lines P_1Q_1, P_2Q_2, and P_3Q_3. Let $|XY|$ mean the length of segment XY. Then $|P_1H|\cdot|HQ_1| = |P_2H|\cdot|HQ_2| = |P_3H|\cdot|HQ_3|$. One must choose P_4 and P_5 on either half line from H perpendicular to plane $P_1P_2P_3$ so that $|P_4H| \neq 0$, $|P_4H| \neq |P_5H|$, and $|P_4H|\cdot|P_5H| = |P_1H|\cdot|HQ_1|$.

B-1. In the additive group of ordered pairs of integers (m, n) [with addition defined componentwise: $(m, n) + (m', n') = (m+m', n+n')$] consider the subgroup H generated by the three elements (3, 8), (4, -1), (5, 4). Then H has another set of generators of the form $(1, b)$, $(0, a)$ for some integers a, b with $a > 0$. Find a.

Ans. The answer is $a = 7$. Also one must have $b \equiv 5 \pmod 7$. Proof: The subgroup H must contain 4(3,8) - 3(4,-1) = (0,35), 4(5,4) - 5(4,-1) = (0,21), and then 2(0,21) - (0,35) = (0,7). Now (0,7) and $(1,b)$ will generate H iff $(1,b)$ is in H and there exist integers u, v, and w such that $(3,8) = 3(1,b) + u(0,7)$, $(4,-1) = 4(1,b) + v(0,7)$, $(5,4) = 5(1,b) + w(0,7)$. These hold iff $8 = 3b + 7u$, $-1 = 4b + 7v$, and $4 = 5b + 7w$. With $b = 5 + 7k$, k any integer, the desired coefficients u, v, and w exist in the form $u = -1 - 3k$, $v = -3 - 4k$, $w = -3 - 5k$. It now suffices to let $k = 0$ and to note that $(1,5) = (4,-1) - (3,8) + 2(0,7)$ is in H.

B-2. In three-dimensional Euclidean space, define a slab to be the open set of points lying between two parallel planes. The distance between the planes is called the thickness of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$, respectively, such that $\Sigma\, d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.

Ans. Let $\Sigma d_i = d$ and let S be a sphere of radius $r > d/2$. The area of S contained in slab S_i is at most $2\pi d_i$. It follows that the area of S contained in the union of the slabs S_i is at most $2\pi d < 4\pi r =$ (area of S). Hence there are points of S that are not in any of the slabs.

B-3. Let $s_k(a_1, \ldots, a_n)$ denote the k-th elementary symmetric function of $a_1, \ldots, a_n$. With k held fixed, find the supremum (or least upper bound) M_k of

$$s_k(a_1, \ldots, a_n)/[s_1(a_1, \ldots, a_n)]^k$$

for arbitrary $n \geq k$ and arbitrary n-tuples of $a_1, \ldots, a_n$ of positive real numbers.

Ans. In the expansion of $s_1^k = (a_1 + a_2 + \ldots + a_n)^k$, every term of s_k appears with $k!$ as coefficient and the other coefficients are nonnegative. Hence $s_k/s_1^k \leq 1/k!$ If we let each $a_i = 1$,

$$\frac{s_k}{s_1^k} = \frac{\binom{n}{k}}{n^k} = \frac{n(n-1)\ldots(n-k+1)}{k!n^k}$$
$$= \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right)$$

which approaches $1/k!$ as k is held fixed and n goes to infinity. These facts show that the supremum M_k is $1/k!$.

B-4. Does there exist a subset B of the unit circle $x^2 + y^2 = 1$ such that (i) B is topologically closed, and (ii) B contains exactly one point from each pair of diametrically opposite points on the circle?

Ans. No. Since the mapping with $(x, y) \to (-x, -y)$ is a homeomorphism of the unit circle on itself, the complement $-B$ of such a subset B would also be closed. Thus the existence of such a B would make C the union $-B \cup C$ of disjoint nonempty closed subsets; this would contradict the fact that C is connected.

B-5. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that $\Sigma\, f_n(1)/n! = e^e$.

Ans. Since $f_0(x) = \Sigma\, x^k/k!$, one easily shows by mathematical induction that $f_n(x) = \Sigma\, k^n x^k/k!$. Then, since all terms are positive, one has

$$\sum_n f_n(1)/n! = \sum_n \sum_k k^n/k!n! =$$

$$\sum_k (1/k!) \sum_n k^n/n! = \sum_k e^k/k! = e^e.$$

B-6. Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, then (a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and (b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

Ans. Both parts are done easily using the inequality on the means. For (a), one has

$$\frac{n+s_n}{n} = \frac{(1+1)+(1+\frac{1}{2})+\ldots+(1+\frac{1}{n})}{n}$$
$$> \sqrt[n]{(1+1)(1+\tfrac{1}{2})\ldots(1+\tfrac{1}{n})} = (n+1)^{1/n}$$

and so $n + s_n > n(n+1)^{1/n}$. For (b), one has

$$\frac{n-s_n}{n-1} = \frac{(1-\frac{1}{2})+(1-\frac{1}{3})+\ldots+(1-\frac{1}{n})}{n-1}$$
$$> \sqrt[n-1]{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\ldots(1-\tfrac{1}{n})} = n^{-1/(n-1)}$$

and so $n-s_n > (n-1)n^{-1/(n-1)}$.

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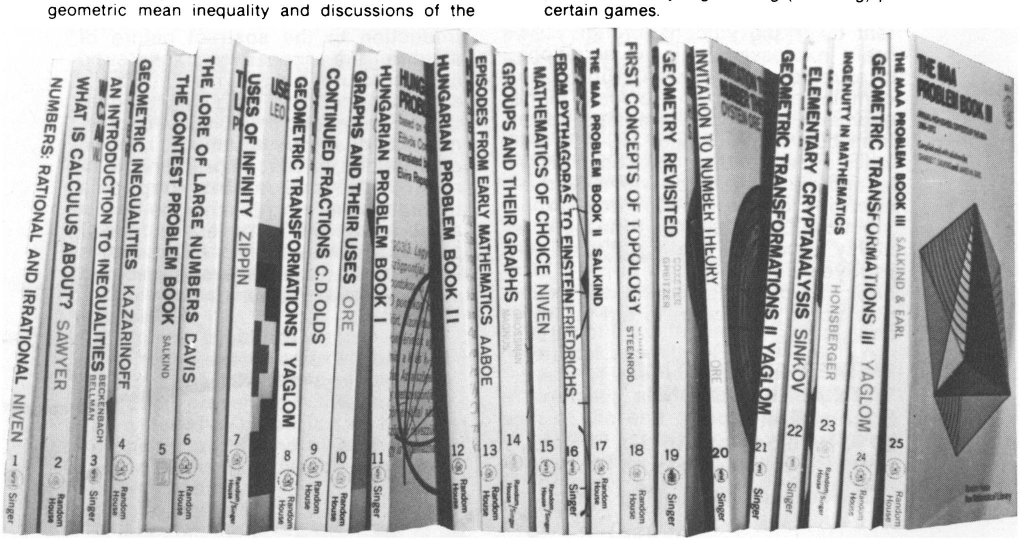

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